Kuwait University
Faculty of Science
Department of Mathematics

Advanced Linear Algebra

0410-363
Final Exam

Sunday, May 19, 2019
Spring 2018/19

Student Name											إسم الطــالب
Student ID Number											الرقم الجامعي للطــالب
											الرقم التسلسلي Serial Number

Section No. رقم الشعبة	Instructor Name أستاذ المقرر
01 A	**Dr. Abdullah Alazemi**

Instructions to students — تعليمات للطالب

Time allowed: 2 hours. — وقت الإختبار : ساعتين.

This exam contains 6 main questions. — يحتوي هذا الإختبار على 6 أسئلة رئيسية.

ممنوع دخول الآلات الحاسبة أو أي وسيلة للإتصال داخل قاعة الإختبار.
Calculators and communication devices are not allowed in the examination room.

Question 1	
Question 2	
Question 3	
Question 4	
Question 5	
Question 6	
Total	

1. (**3+3 pts.**) Let $\mathbb{V}$ be an inner product space over a field $\mathbf{F}$. Show that:

(a) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$, for any $x, y, z \in \mathbb{V}$.

(b) For $x = (1, i)$, $y = (2, i)$ in $\mathbb{C}^2$, define $\langle x, y \rangle = x\,A\,y^*$, where $A = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$. Compute $\langle x, y \rangle$.

2. (**4+2 pts.**) Let β be a basis for a finite-dimensional inner product space $\mathbb{V}$.

(a) Show that if $\langle x, z \rangle = 0$ for all $z \in \beta$, then $x = 0$.

(b) Show that if $\langle x, z \rangle = \langle y, z \rangle$ for all $z \in \beta$, then $x = y$.

3. (**3+3 pts.**)

(a) Let $\mathbf{T}$ be the linear operator on $\mathbb{C}^2$ defined by $\mathbf{T}(a, b) = (ai + b, a - b)$. Evaluate $\mathbf{T}^*$ at $x = (1, i)$.

(b) Let $\mathbb{V}$ be an inner product space, and let $y, z \in \mathbb{V}$. Define the operator $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ by $\mathbf{T}(x) = \langle x, y \rangle z$ for all $x \in \mathbb{V}$. Evaluate $\mathbf{T}^*(x)$.

4. (**3+3+4 pts.**) Let $\mathbf{T}$ be a linear operator on a finite-dimensional inner product space $\mathbb{V}$ over a field $\mathbf{F}$. Then:

(a) If $\mathbf{T}$ is normal and λ_1 and λ_2 are two distinct eigenvalues of $\mathbf{T}$ with corresponding eigenvectors x_1 and x_2, respectively, then x_1 and x_2 are orthogonal.

(b) If $\mathbf{T}$ is self-adjoint, then every eigenvalue of $\mathbf{T}$ is real.

(c) If $\mathbf{T}\mathbf{T}^* = \mathbf{T}^*\mathbf{T} = \mathbf{I}_V$ and β is an orthonormal basis for $\mathbb{V}$, then $\mathbf{T}(\beta)$ is an orthonormal basis for $\mathbb{V}$.

5. (**6 pts.**) Let $\mathbb{V} = \mathbb{P}_1(\mathbb{R})$ with an inner product defined by $\langle f(x), g(x) \rangle = \int_0^1 f(x)g(x)\,dx$. Use the Gram-Schmidt process to replace the standard ordered basis $S = \{1, x\}$ by an orthonormal basis for $\mathbb{P}_1(\mathbb{R})$. Represent $h(x) = 1 + 2x$ as a linear combination of the vectors of the obtained orthonormal basis for $\mathbb{P}_1(\mathbb{R})$.

6. (**6 pts.**) Let $\mathbf{T}$ be an operator on $\mathbb{P}_1(\mathbb{R})$ defined by $\mathbf{T}(f) = f'$, where $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$. Determine whether $\mathbf{T}$ is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of $\mathbf{T}$ for $\mathbb{V}$.