1. (2+2+4 pts.)

(a) Let $\mathbb{W}$ be the set of all nonsingular matrices in $M_{n\times n}(\mathbb{R})$. Decide whether $\mathbb{W}$ is a subspace of $M_{n\times n}(\mathbb{R})$.

(b) Let $\mathbb{W} = \{\, a + bx + cx^2 : a = b = c\,\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{P}_2(\mathbb{R})$.

(c) Let $\mathbf{T}$ be the linear operator on $\mathbb{R}^2$ defined by $\mathbf{T}\,(x, y) = (\,2x + y, x - y\,)$. Find $\mathcal{N}(\mathbf{T})$, $\mathcal{R}(\mathbf{T})$, nullity$(\mathbf{T})$, rank$(\mathbf{T})$, Decide whether $\mathbf{T}$ is a bijection.

2. (2+3+3 pts.)

(a) Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$. Assume that $\mathbf{T} : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_1(\mathbb{R})$ is the linear defined by A using the standard ordered basis β and γ for $\mathbb{P}_2(\mathbb{R})$ and $\mathbb{P}_1(\mathbb{R})$, respectively. Evaluate $\mathbf{T}(g(x))$, where $g(x) = 2x^2 - 3x + 1$.

(b) Let $\mathbf{T}$ be the operator on $\mathbb{P}_1(\mathbb{R})$ defined by $\mathbf{T}(f(x)) = f'(x)$. Is $\mathbf{T}$ diagonalizable? Explain.

(c) Let $\mathbf{T}$ be a linear operator on a finite-dimensional inner product space $\mathbb{V}$ so that for all $x, y \in \mathbb{V}$, $\langle \mathbf{T}(x), \mathbf{T}(y) \rangle = \langle x, y \rangle$. Show that if β is an orthonormal basis for $\mathbb{V}$, then so is $\mathbf{T}(\beta)$.

3. (**4+4 pts.**)

(a) Let $\mathbb{V}$ be an inner product space, and let $y, z \in \mathbb{V}$. Define $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ by $\mathbf{T}(x) = \langle x, y \rangle z$ for all $x \in \mathbb{V}$. Show that $\mathbf{T}$ is linear, and evaluate $\mathbf{T}^*(x)$.

(b) Let $\mathbf{T}$ be the linear operator on $\mathbb{C}^2$ defined by $\mathbf{T}(x, y) = (2xi + 3y, x - y)$. Evaluate $\mathbf{T}^*(1, i)$.

4. (4+4 pts.)

(a) Let $\mathbb{W} = \{\, (x+y, x, x+2y) : x, y \in \mathbb{R} \,\}$ be a set in $\mathbb{R}^3$. Show that $\mathbb{W}$ is a subspace of $\mathbb{R}^3$ and find an orthonormal basis for $\mathbb{W}$.

(b) Let $\mathbb{V}$ be an inner product space, and let $S = \{\, x_1, x_2, \cdots, x_n \,\}$ be an orthonormal basis for $\mathbb{V}$. Show that $y = \sum_{i=1}^{n} \langle\, y, x_i \,\rangle\, x_i$, for any $y \in \mathbb{V}$.

5. (5+5 pts.)

(a) Let $\mathbf{T}$ be the linear operator on $\mathbb{P}_1(\mathbb{R})$ defined by $\mathbf{T}(f) = f'$, where $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$. Determine whether $\mathbf{T}$ is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of $\mathbf{T}$ for $\mathbb{P}_1(\mathbb{R})$.

(b) Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Show that A is orthogonally equivalent to a diagonal matrix, and find a diagonal matrix D such that $P^t A P = D$, for some orthogonal matrix P. [**You do not need to compute** P].