Kuwait University
Faculty of Science
Department of Mathematics

Euclidean Geometry

0410-226
Second Exam

Monday, November 25, 2019
Fall 2019/2020

Name										
ID Number										
Serial Number										

Duration **75** minutes (This exam contains **4** questions).

Section No.	Instructor Name
1	**Dr. Abdullah Alazemi**

Calculators and communication devices are not allowed in the examination room.
Give full reasons for your answer and State clearly any Theorem you use.

Question 1	
Question 2	
Question 3	
Question 4	
Total	

1. (**2 pts. each**) Let $\mathbf{T}$ and $\mathbf{S}$ be two isometries of the plane.

(a) Show that the product of $\mathbf{T}$ and $\mathbf{S}$ is an isometry.

(b) Show that if $\mathbf{T}$ and $\mathbf{S}$ agrees on three noncollinear points, then they are identical.

<u>**Solution**</u>:

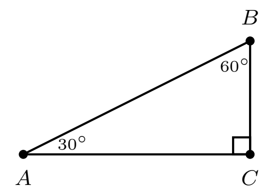
(a) For any A and B, $\mathbf{T}\left(\overline{AB}\right) = \overline{A'B'}$ and $\mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, where $\overline{AB} \cong \overline{A'B'}$ ($\mathbf{T}$ is isometry) and $\overline{A'B'} \cong \overline{A''B''}$ ($\mathbf{S}$ is isometry).

Therefore, $\mathbf{ST}\left(\overline{AB}\right) = \mathbf{S}\left(\mathbf{T}\left(\overline{AB}\right)\right) = \mathbf{S}\left(\overline{A'B'}\right) = \overline{A''B''}$, with $\overline{AB} \cong \overline{A''B''}$.

(b) Assume that $\mathbf{T}(A) = \mathbf{S}(A)$, $\mathbf{T}(B) = \mathbf{S}(B)$, $\mathbf{T}(C) = \mathbf{S}(C)$, for noncollinear points A, B, C. Then $\mathbf{S}^{-1}\mathbf{T}(A) = A$, $\mathbf{S}^{-1}\mathbf{T}(B) = B$, $\mathbf{S}^{-1}\mathbf{T}(C) = C$. That is $\mathbf{S}^{-1}\mathbf{T} = \mathbf{I}$. Hence, $\mathbf{T} = \mathbf{S}$.

2. (**3 pts. each**)

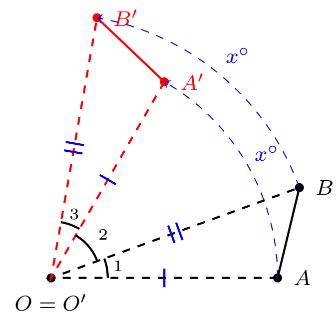
(a) Show that a rotation is an isometry.



(b) In the right diagram, identify $\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathcal{R}_{B,120^\circ} \circ \mathbf{R}_{\overleftrightarrow{AC}}$.

(c) In the right diagram, express the rotation $\mathcal{R}_{B,240^\circ}$ as a product of two rotations centered at A and C.

<u>**Solution**</u>:



(a) Consider a rotation $\mathcal{R}_{O,x}$ about some point O through x°. Let A and B be points in the plane with $\mathcal{R}_{O,x}(A) = A'$ and $\mathcal{R}_{O,x}(B) = B'$. Then, we need to show that $\left|\overline{AB}\right| = \left|\overline{A'B'}\right|$. In $\triangle AOB$ and $\triangle A'OB'$, we have:

i. $\left|\overline{OA}\right| = \left|\overline{OA'}\right|$ and $\left|\overline{OB}\right| = \left|\overline{OB'}\right|$ (definition of rotation).

ii. $\left|\hat{1}\right| = x - \left|\hat{2}\right| = \left|\hat{3}\right|$ (look at diagram).

By SAS, $\triangle AOB \cong \triangle A'OB'$. That is $\left|\overline{AB}\right| = \left|\overline{A'B'}\right|$.

(b) Note that $\mathcal{R}_{B,120^\circ} = \mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AB}}$. Therefore,

$$\mathbf{R}_{\overleftrightarrow{BC}} \circ \mathcal{R}_{B,120^\circ} \circ \mathbf{R}_{\overleftrightarrow{AC}} = \mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{BC}} \circ \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}} = \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}} = \mathcal{R}_{A,60^\circ}.$$

(c) Note that $\mathcal{R}_{B,240^\circ} = \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{BC}}$. Therefore,

$$\begin{aligned}
\mathcal{R}_{B,240^\circ} &= \mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{BC}} = \mathbf{R}_{\overleftrightarrow{AB}} \circ I \circ \mathbf{R}_{\overleftrightarrow{BC}} \\
&= \mathbf{R}_{\overleftrightarrow{AB}} \circ \left(\mathbf{R}_{\overleftrightarrow{AC}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right) \circ \mathbf{R}_{\overleftrightarrow{BC}} \\
&= \left(\mathbf{R}_{\overleftrightarrow{AB}} \circ \mathbf{R}_{\overleftrightarrow{AC}}\right)\left(\circ\, \mathbf{R}_{\overleftrightarrow{AC}} \circ \mathbf{R}_{\overleftrightarrow{BC}}\right) \\
&= \mathcal{R}_{A,60^\circ} \circ \mathcal{R}_{C,180^\circ}.
\end{aligned}$$

3. (**4** + **2 pts.**)

(a) Show that every translation is a product of two half-turns.

(b) Let $\mathbf{T}$ be a translation identified as $\mathbf{R}_a \circ \mathbf{R}_b$. Find its inverse.

Solution:

(a) Let $\mathcal{T}_{\overrightarrow{AB}}$ be any translation. Then $\mathcal{T}_{\overrightarrow{AB}}$ can be written as a product of two reflections in parallel lines a and b. That is, $\mathcal{T}_{\overrightarrow{AB}} = \mathbf{R}_a\mathbf{R}_b$. Let c be a line perpendicular to a and b in points O_1 and O_2. Then, $\mathcal{H}_{O_1} = \mathbf{R}_a\mathbf{R}_c$ and $\mathcal{H}_{O_2} = \mathbf{R}_c\mathbf{R}_b$. Therefore,

$$\mathcal{T}_{\overrightarrow{AB}} = \mathbf{R}_a\mathbf{R}_b = \mathbf{R}_a\mathbf{I}\mathbf{R}_b = \mathbf{R}_a\mathbf{R}_c\mathbf{R}_c\mathbf{R}_b = \mathcal{H}_{O_1}\mathcal{H}_{O_2}.$$

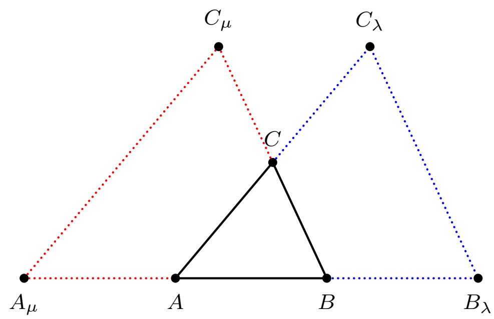
(b) Clearly $\mathbf{T}^{-1} = \mathbf{R}_b \circ \mathbf{R}_a$.

4. **(4+2 pts.)** Let $\triangle\, ABC$ be a given acute triangle and let $\lambda, \mu > 1$.

(a) Show that $\mathcal{D}_{A,\lambda}(\triangle\, ABC) \sim \mathcal{D}_{B,\mu}(\triangle\, ABC)$.

(b) Show that if $\lambda = \mu$, then $\mathcal{D}_{A,\lambda}(\triangle\, ABC) \cong \mathcal{D}_{B,\mu}(\triangle\, ABC)$.

Solution:



(a) Clearly **Theorem 5.1.1** ensures that the trangles $\triangle\, AB_\lambda C_\lambda$ and $\triangle\, A_\mu BC_\mu$ are all similar to $\triangle\, ABC$ and hence each one of them is similar to the other.

(b) If $\lambda = \mu$ (both positive), then we have $\left|\overline{A_\mu B}\right| = \mu\left|\overline{AB}\right| = \lambda\left|\overline{AB}\right| = \left|\overline{AB_\lambda}\right|$. Similarily, $\left|\overline{A_\mu C_\mu}\right| = \left|\overline{AC_\lambda}\right|$ and $\left|\overline{BC_\mu}\right| = \left|\overline{B_\lambda C_\lambda}\right|$. By SSS, $\triangle\, AB_\lambda C_\lambda \cong \triangle\, A_\mu BC_\mu$.