



Kuwait University
Faculty of Science
Department of Mathematics

Euclidean Geometry 0410-226

Final Exam

Monday, December 16, 2019
Fall 2019/2020

Name										
ID Number										
Serial Number										

Duration 2 hours (This exam contains 6 questions).

Section No.	Instructor Name
1	Dr. Abdullah Alazemi

Calculators and communication devices are not allowed in the examination room.

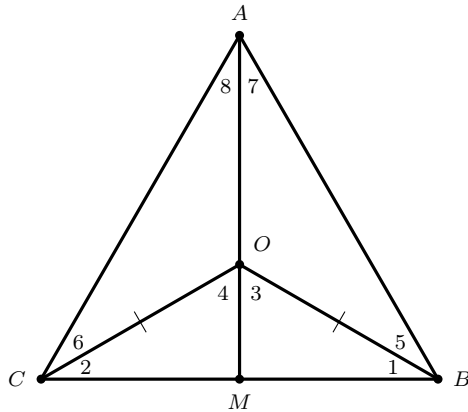
Give full reasons for your answer and State clearly any Theorem you use.

Question 1	/	6
Question 2	/	9
Question 3	/	6
Question 4	/	10
Question 5	/	9
Question 6	/	2
Total	/	42

Good Luck

1. (6 pts.) In the diagram, let \overline{AM} bisect the angle $\hat{C}OB$. Assume that $\triangle BCO$ is an isosceles triangle. Show that $\triangle ABM \cong \triangle ACM$.

Solution:



In triangles $\triangle OBM$ and $\triangle OCM$, we have:

- (a) $\hat{3} \cong \hat{4}$ (\overline{AM} is a bisector).
- (b) $\overline{OB} \cong \overline{OC}$ (given - $\triangle BCO$ is isosceles).
- (c) $\hat{1} \cong \hat{2}$ (given - $\triangle BCO$ is isosceles).

By ASA, we have $\triangle OBM \cong \triangle OCM$. That is $\overline{BM} \cong \overline{CM}$ and $\hat{BMA} \cong \hat{CMA}$ (both right angles). Now, in triangles $\triangle ABM$ and $\triangle ACM$, we have:

- (a) \overline{AM} - common.
- (b) $\hat{AMB} \cong \hat{AMC}$ (proved).
- (c) $\overline{MB} \cong \overline{MC}$ (proved).

By SAS, we have $\triangle ABM \cong \triangle ACM$.

2. (3 pts. each)

- (a) Let \overline{RS} be tangent to $\odot A$ and $\odot B$. Show that $\triangle ARC \sim \triangle BSC$.
 (b) If $|\overline{BS}| = 0.5|\overline{AR}|$, define a transformation that maps $\odot B$ to $\odot A$.
 (c) Let \overline{AB} be a diameter of the circle $\odot O$, $|\widehat{BC}| = 40^\circ$, and $|\hat{1}| = 60^\circ$. Find $|\widehat{BD}|$.

Solution:

Figure: Part (a) and (b)

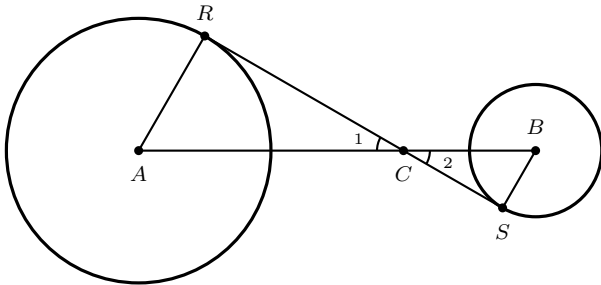
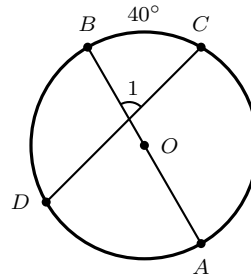


Figure: Part (c)



- (a) Clearly, $\hat{1} \cong \hat{2}$ (vertically opposite). Also $\hat{S} \cong \hat{R}$ (both are right angles). By S-AA, $\triangle ARC \sim \triangle BSC$.
 (b) A dilation $\mathcal{D}_{C,-2}$ would map $\odot B$ to $\odot A$.
 (c) Clearly, by Theorem in class $|\hat{1}| = \frac{1}{2}(|\widehat{BC}| + |\widehat{AD}|)$. That is $60^\circ = \frac{1}{2}(40^\circ + |\widehat{AD}|)$.
 Hence $|\widehat{AD}| = 120^\circ - 40^\circ = 80^\circ$. Therefore, $|\widehat{BD}| = 100^\circ$

3. (3 pts. each)

- (a) Let O , P and Q be three noncollinear points so that $\mathcal{R}_{O,\theta}(O) = O$, $\mathcal{R}_{O,\theta}(P) = P'$ and $\mathcal{R}_{O,\theta}(Q) = Q'$. Show that $\overline{PQ} \cong \overline{P'Q'}$.
- (b) Assume that $\triangle ABC$ and $\triangle DEC$ are two isosceles right triangles. Show that $\overline{AD} \cong \overline{BE}$.

Solution:

Figure: Part (a)

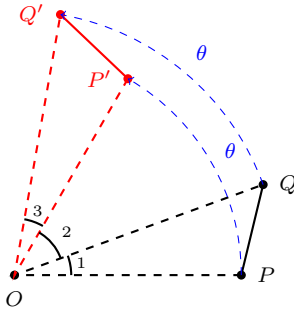
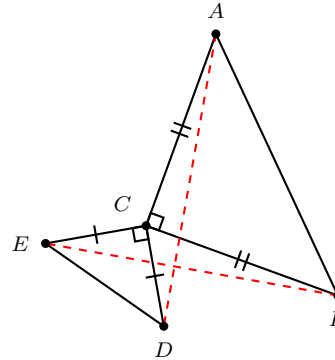


Figure: Part (b)



- (a) We need to show that $|\overline{PQ}| = |\overline{P'Q'}|$. In $\triangle POQ$ and $\triangle P'OQ'$, we have:

- i. $|\overline{OP}| = |\overline{OP'}|$ and $|\overline{OQ}| = |\overline{OQ'}|$ (definition of rotation).
- ii. $|\hat{1}| = \theta - |\hat{2}| = |\hat{3}|$ (look at diagram).

By SAS, $\triangle POQ \cong \triangle P'OQ'$. That is $|\overline{PQ}| = |\overline{P'Q'}|$.

- (b) Note that $\mathcal{R}_{C,90}(B) = A$ and $\mathcal{R}_{C,90}(E) = D$. Hence, under the same rotation, we have \overline{BE} maps to \overline{AD} . That is $\mathcal{R}_{C,90}(\overline{BE}) = \overline{AD}$. Therefore, $\overline{AD} \cong \overline{BE}$ since rotation is an isometry (or by part a).

4. (4+3+3 pts.)

- (a) Let $l_1 : x + y - 1 = 0$ and $l_2 : x - y - 1 = 0$ be two lines. Find all points that are equidistant from l_1 and l_2 .
- (b) Use lines l_1 and l_2 (from part a) to write the product $\mathbf{R}_{l_1} \circ \mathbf{R}_{l_2}$ as a single rotation centered at the intersection point (if any) of the two lines.
- (c) Find an equation of the circle with center $A(-1, 2)$ and tangent to the y -axis. Furthermore, **if any** find the point(s) of intersection of the circle and x -axis.

Solution:

- (a) Let $M(x, y)$ be the points of the locus. Thus,

$$d(M, l_1) = d(M, l_2) \Rightarrow \frac{|x + y - 1|}{\sqrt{1 + 1}} = \frac{|x - y - 1|}{\sqrt{1 + 1}} \Rightarrow |x + y - 1| = |x - y - 1|.$$

That is we have two cases:

Case 1: $(x + y - 1) = +(x - y - 1)$, and hence $y = 0$.

Case 2: $(x + y - 1) = -(x - y - 1)$, and hence $x = 1$.

Therefore, all points that are equidistant from l_1 and l_2 lie on the two lines $y = 0$ and $x = 1$

- (b) Note that the slopes of l_1 and l_2 are -1 and 1 , respectively. Thus, the two lines are perpendicular. Thus the angle between the two lines is 90° . Moreover, the two lines are intersecting in the point $(1, 0)$. Therefore, $\mathbf{R}_{l_1} \circ \mathbf{R}_{l_2} = \mathcal{R}_{(1,0),180^\circ}$.

- (c) Since the circle is tangent to y -axis, we have the radius equals to the x -coordinates of A which is the distance from y -axis to A . Thus, $r = |-1| = 1$, and hence the circle equation: $(x + 1)^2 + (y - 2)^2 = 1$.

Substitute $y = 0$ in the circle equation to get $(x + 1)^2 = -4$. This is not possible. Hence, there are no intersection points.

5. (3 pts. each) Find the image of the circle $(x - 1)^2 + (y - 3)^2 = 1$ under each of the following:

(a) A reflection in the line $y - 3x = 0$.

(b) A rotation $\mathcal{R}_{(1,-1),\frac{\pi}{2}}$.

(c) A dilation $\mathcal{D}_{O,-3}$.

Solution:

(a) Note that the center of the circle $P(1, 3)$ lies on the line of reflection. Therefore, the circle is invariant, and hence the image is again: $(x - 1)^2 + (y - 3)^2 = 1$.

(b) The rotation matrix about the origin through the angle $\frac{\pi}{2}$ is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. But we are rotating about $(1, -1)$. Thus the image of $P(1, 3)$ is $P'(x, y)$ where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

Therefore, the image of the circle is: $(x + 3)^2 + (y + 1)^2 = 1$.

(c) Dilation is **not** an isometry. Hence the new radius $r' = |\lambda|1 = |-3| = 3$.

Also, $\mathcal{D}_{O,-3}(A(1, 3)) = A'(-3, -9)$. Thus,

$$c'(A', r') : (x + 3)^2 + (y + 9)^2 = 9.$$

