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1. Let S denote any nonempty set. Show that the composition is an associative operation on $M(S)$, the set of all mappings on S, and show that the identity mapping I_S is the identity element in $M(S)$.

2. Let G be a group with operation . Show that the identity element of G is unique.

3. Show that $G = \{2^m \cdot 3^n \ : \ m, n \in \mathbb{Z}\}$ is a group with respect to multiplication.

4. Find $x \in S_6$ so that: $(1\ \ 2\ \ 3)^{-1}\ x\ (4\ \ 5\ \ 6)(1\ \ 3\ \ 2)(4\ \ 5\ \ 6)^{-1} = (1)$.

5. Let $S = \{1, 2, 3, 4, 5\}$, $G = S_5$, and $T = \{5\}$. Find G_T, the element-wise stabilizer of T in G.

6. Let H and K be two subgroups of a group G. Show that $H \cap K$ is also a subgroup of G.

7. Let a be an element of a group G. Show that $C(a)$, the centralizer of a in G, is a subgroup of G. Find $C(e)$.

8. Let H and K be two subgroups of an abelian group G. Show that $HK = \{hk : h \in H \text{ and } k \in K\}$ is also a subgroup of G.

9. Let n be an integer so that $n > 1$.

   (a) Show that $\mathbb{U}_n$ is closed under the operation $\odot$.

   (b) Use the Euclidean algorithm to compute the greatest common divisor of 13 and 40, and write it as a linear combination of 13 and 40.

   (c) What is the inverse of [13] in $\mathbb{U}_{40}$.

   (d) Find the order of $\mathbb{U}_{40}$.

   (e) Find, if possible, the inverse of [12] in $\mathbb{U}_{40}$.

10. Show that every non-abelian group is non-cyclic.

11. Show that every group of even order has an element of order 2.

12. Let G be a group with no subgroup other than G and $\{e\}$. Show that G is cyclic.

13. Simplify: $\big([2],(1\ \ 2\ \ 3)\big)^{-1}\big([1],(2\ \ 4)\big)\big([2],(1\ \ 2\ \ 3)\big)$ in $\mathbb{Z}_4 \times S_4$.

14. Let G and H be two groups. Show that $G \times \{e_H\}$ is a subgroup of $G \times H$.

15. Let H be a subgroup of a group G and define a relation $\sim$ on G by $a \sim b$ iff $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

16. Define a relation $\sim$ on the set $\mathbb{N}$ of natural numbers by $a \sim b$ iff $a = b \cdot 10^k$ for some $k \in \mathbb{Z}$. Show that $\sim$ is an equivalence relation on $\mathbb{N}$

17. Compute the distinct left cosets of $H = \langle(1\ \ 2)\rangle \times \langle 1\rangle$ in $S_3 \times \mathbb{Z}_2$.

18. Show that a group G of a prime order contains no subgroups other than G and $\{e\}$.

19. Let G be a non-abelian group of order 22. Show that G has an element of order 11.

20. List all subgroups of $\mathbb{Z}_{12}$.

21. Let G be a non-abelian group of order 49. Show that $a^7 = e$ for each $a \in G$.

22. Let H be a subgroup of index 2 in a group G. Show that H is a normal subgroup of G (That is show that $aH = Ha$ for each $a \in G$).

23. Let G and H be two isomorphic groups. Show that if G is cyclic, then H is cyclic as well.

24. Let G and H be two groups. If $\theta : G \to H$ is a homomorphism mapping onto H, then $\theta(G) = \big\{\theta(g) : g \in G\big\}$ is a subgroup of H.

25. Let $\theta : (\mathbb{R}, +) \to (\mathbb{R}^+, \dot{})$ defined by $\theta(x) = e^x$ for each $x \in \mathbb{R}$. Show that θ is an isomorphism.

26. Show that the isomorphism relation, denoted by $\sim$, is an equivalence relation on the class of all groups.

27. Show that every group G of a prime order p is isomorphic to $\mathbb{Z}_p$.

28. List the isomorphism class representatives of abelian groups of order 150.

29. For an integer $n > 1$, define $\theta : \mathbb{Z} \to \mathbb{Z}_n$ by $\theta(a) = [a]$ for each $a \in \mathbb{Z}$.

    (a) Show that θ is a homomorphism onto $\mathbb{Z}_n$.

    (b) Find $\ker\theta$.

    (c) Use the fundamental homomorphism theorem to show that $\mathbb{Z}/ < n >$ is isomorphic to $\mathbb{Z}_n$.

30. Let $\theta : G \to H$ be a group homomorphism.

    (a) Show that θ is one-to-one if and only if $\ker\theta = \{e_G\}$.

    (b) Show that $\ker\theta$ is normal subgroup of G.

31. Show that every quotient group of a cyclic group is cyclic.

32. Prove that if M and N are normal subgroups of a group G, then $M \cap N$ is a normal subgroup of G.

33. Show that if M and N are normal subgroups of a group G and $M \cap N = \{e\}$, then $mn = nm$ for all $m \in M$ and for all $n \in N$.

34. Let H and K be two subgroups of a finite group G. If $\text{GCD}(|H|, |K|) = 1$, then $H \cap K = \{e\}$.