Revision for Second Exam

1. Let $D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 5 & 8 & 9 \\ 3 & 4 & 5 & 10 \\ 4 & 9 & 13 & 2013 \end{bmatrix}$. Evaluate $\left|D\right|$.

   [**Hint**: $|D| = 0$.]

2. Given that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 7$, evaluate $\begin{vmatrix} b_1 & b_2 & b_1 - 3b_3 \\ a_1 & a_2 & a_1 - 3a_3 \\ c_1 & c_2 & c_1 - 3c_3 \end{vmatrix}$.

   [**Hint**: The result is 21.]

3. Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$, and $B = \begin{bmatrix} 2a_3 & 2a_2 & 2a_1 \\ b_3 - a_3 & b_2 - a_2 & b_1 - a_1 \\ c_3 + 3b_3 & c_2 + 3b_2 & c_1 + 3b_1 \end{bmatrix}$. If $\left|A\right| = -4$, find $\left|B\right|$.

   [**Hint**: The result is 8.]

4. Let A be a matrix with $A^{-1} = \begin{bmatrix} 7 & 1 & 0 & 3 \\ 2 & 0 & 0 & 0 \\ 1 & 3 & 5 & 4 \\ 6 & 2 & 0 & 5 \end{bmatrix}$. Find $\det(A)$.

5. Find all values of α for which the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & \alpha & \alpha^2 \end{bmatrix}$ is singular.

6. Let A and B be two $n \times n$ matrices such that A is invertible and B is singular. Prove that $A^{-1}B$ is singular.

7. If A and B are 2×2 matrices with $\det(A) = 2$ and $\det(B) = 5$, compute $\left|3\,A^2(AB^{-1})^T\right|$.

8. Let $\operatorname{adj}(A) = \begin{bmatrix} 2 & 8 \\ 1 & 4 \end{bmatrix}$. Find A.

9. Let A be a nonsingular $n \times n$ matrix.

   (a) Show that $\operatorname{adj}(A)$ is nonsingular.

   (b) Prove that $\operatorname{adj}(A^{-1}) = \frac{1}{\det(A)} A$.

10. Find all 3×3 matrices A, if any, such that $\left|\operatorname{adj}(A)\right| = -3$.

11. Let A and B be two 4×4 nonsingular matrices such that $\det(2A^{-1}B^2) = 32$ and $\det(B^{-1}\operatorname{adj}(A)) = 4$. Find $|A|$ and $|B|$.

12. Let A be a nonsingular $n \times n$ matrix. Prove that $\det(\operatorname{adj}(A)) = [\det(A)]^{n-1}$.

13. Show that if A is a nonsingular matrix, then $\operatorname{adj}(A)$ is nonsingular and

$$(\operatorname{adj}(A))^{-1} = \frac{1}{\det(A)} A = \operatorname{adj}(A^{-1}).$$

14. Let $A = \begin{bmatrix} 1 & 2 & -2 & 3 \\ 0 & -1 & 3 & 1 \\ 0 & 2 & -6 & 1 \\ 2 & 4 & -3 & 0 \end{bmatrix}$. Find $\det(A)$ by:

   (a) reducing A to triangular form.

   (b) using a cofactor expansion of $\det(A)$.

   (c) using the cofactor expansion on the fourth row of A.

15. Let A be a matrix with $\operatorname{adj}(A) = \frac{1}{2} \begin{bmatrix} 0 & 0 & 6 & 0 \\ 3 & 0 & 7 & 0 \\ 1 & 4 & 8 & 0 \\ 7 & 1 & 9 & 6 \end{bmatrix}$. Find A^{-1}.

16. Find $\operatorname{adj}(A)$, given that $A^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 7 & 6 & 4 \\ 4 & 4 & 4 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$.

17. Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 5 & -3 & 6 \\ 2 & 3 & 2 \end{bmatrix}$. Compute the cofactors A_{11}, A_{12}, and A_{13}, and show that $5A_{11} - 3A_{12} + 6A_{13} = 0$.

18. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

(a) Compute $\det(A)$ using a cofactor expansion along row 2.

(b) Find $A\,\text{adj}(A)$.

(c) Find $\det(2A^{-1}\text{adj}(A))$.

19. Let A be an $n \times n$ non-singular matrix. Show that if A is symmetric, then $\text{adj}(A)$ is symmetric.

20. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \\ -1 & 3 & 1 \end{bmatrix}$.

(a) Find $|A|$ using a cofactor expansion along row 2.

(b) Find $A\,\text{adj}(A)$.

(c) Find $|2A^{-1}adj(A)|$.

21. Answer each of the following as True or False (Justify your answer):

(a) (F) If A and B are 2×2 matrices such that $\det(A) = -2$ and $\det(B) = 8$, then $\det\left(2A^T\text{adj}(B^{-1})\right) = 1$.

(b) (F) if A and B are two 2×2 matrices such that $\det(\text{adj}(A)) = \det(\text{adj}(B))$, then $A = B$.

22. Use Cauchy-Schwarzs inequality to show that

$$(ab - cd + xy)^2 \leq (a^2 + d^2 + y^2)(b^2 + c^2 + x^2)$$

for all real numbers a, b, c, d, x, and y.

23. Find all vectors in $\mathbb{R}^4$ which are perpendicular to the vectors $X = (1, 1, 2, 2)$ and $Y = (2, 3, 5, 5)$.

24. Let $U, V \in \mathbb{R}^n$ be unit vectors. Prove that $(U + 2V) \cdot (2U - V) \leq 3$.

25. Verify that the triangle with vertices $A(1, 1, 2)$, $B(1, 2, 3)$, and $C(3, 0, 3)$ is a right triangle.

26. Let θ be the angle between the vectors $U = (4, 2, 1, 2)$ and $V = (4, 2, 5, 2)$. Find $\cos\theta$.

27. For any vectors X and Y in $\mathbb{R}^n$, show that $\|X\| \leq \|X - 2Y\| + 2\|Y\|$.

28. Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in $\mathbb{R}^n$. Prove that $\|\mathbf{x} - \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

29. Find all values of a for which $X = (a^2 - a, -3, -1)$ and $Y = (2, a - 1, 2a)$ are othogonal.

30. Find a vector X, of length 6, in the opposite direction of $Y = (1, 2, -2)$.

31. Let X and Y be vectors in $\mathbb{R}^n$ such that $\|X\| = \|Y\|$. Show that $X + Y$ and $X - Y$ are orthogonal vectors.

32. Find all $x, y \in \mathbb{R}^3$ such that the vectors $U = (x, 1, 3)$ and $V = (4, x + y, 9)$ are parallel. [Hint: U and V are parallel iff there is $r \in \mathbb{R}^3$ such that $U = rV$.]

33. Let U and V be two vectors in $\mathbb{R}^3$ such that $\|U\| = 2$ and $\|V\| = 3$.

    (a) Find the maximum possible value for $\|2U + 3V\|$.

    (b) If U and V are orthogonal, find $\|2U + 3V\|$.

34. Let $X, Y \in \mathbb{R}^n$. Find $X \cdot Y$ given that $\|X + Y\| = 1$ and $\|X - Y\| = 5$.

35. Answer each of the following as True or False:

    (a) (T) If U and V are two unit vectors in $\mathbb{R}^n$, then $\|U - 6V\| \geq 5$.

    (b) (F) There exist $X, Y \in \mathbb{R}^4$ such that $\|X\| = \|Y\| = 2$ and $X \cdot Y = 6$.

36. Show that two nonzero vectors X and Y in $\mathbb{R}^3$ are parallel, if and only if, $X \times Y = 0$.

37. If U and V are nonzero vectors in $\mathbb{R}^3$ such that $\|(2U) \times (2V)\| = 4U \cdot V$, compute the angle between U and V.

38. Find the area of the triangle whose vertices are $P(1, 0, -1)$, $Q(2, -1, 3)$ and $R(0, 1, -2)$.

39. Let U and V be unit vectors in $\mathbb{R}^3$. Show that $\|U \times V\|^2 + (U \cdot V)^2 = 1$.

40. Let X and Y be two nonzero vectors in $\mathbb{R}^3$, with angle $\theta = \frac{\pi}{3}$ between them. Find $\|X \times Y\|$, if $\|X\| = 3$ and $\|-2Y\| = 4$.

41. If X and Y are two vectors in $\mathbb{R}^3$, show that $X \times Y$ is orthogonal to X.

42. Consider the planes:

$$\mathbf{\Pi}_1 : \ x + y + z = 3, \quad \mathbf{\Pi}_2 : \ x + 2y - 2z = 2k, \quad \text{and } \mathbf{\Pi}_3 : \ x + k^2 z = 2.$$

Find all values of k for which the intersection of the three planes is a line.

[**Hint:** Any point on the intersection of the three planes must satisfies the three equations. This would give a system of three equation. This system must have infinitely many solutions to describe a line.]

43. Consider the lines:

$$\boldsymbol{L}_1 : \ \frac{x-7}{2} = \frac{y+3}{-1} = z - 1, \quad \text{and} \quad \boldsymbol{L}_2 : \ x + 1 = \frac{y+6}{3} = \frac{z-2}{-2}.$$

(a) Show that $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ intersect and find their point of intersection.

(b) Find parametric equations of $\boldsymbol{L}_3$ which is perpendicular to both $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ and passes through their point of intersection.

[**Hint:** (a) Show that $\boldsymbol{U}_1 \times \boldsymbol{U}_2 \neq \mathbf{0}$. Point of intersection is $P(1, 0, -2)$. (b) Consider $\boldsymbol{U}_3 = \boldsymbol{U}_1 \times \boldsymbol{U}_2$ which is orthogonal to both $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$.]

44. Consider the lines:

$$\boldsymbol{L}_1 : \ \frac{x+1}{3} = \frac{y+4}{2} = z - 1, \quad \text{and} \quad \boldsymbol{L}_2 : \ \frac{x-3}{2} = \frac{y-4}{-4} = \frac{z-2}{2}.$$

(a) Show that $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are perpendicular and find their point of intersection.

(b) Find an equation of the plane $\mathbf{\Pi}$ that contains both $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$.

[**Hint:** (a) Show that $\boldsymbol{U}_1 \cdot \boldsymbol{U}_2 = 0$ and then find a point satisfying both equations of x, y, and z in terms of t_1 and t_2, for instance. (b) Consider $\boldsymbol{N} = \boldsymbol{U}_1 \times \boldsymbol{U}_2$.]

45. Find an equation of the plane $\mathbf{\Pi}$ through the point $P(0, 1, 1)$ and containing the line $x = 1 + t$, $y = 1 + 2t$, and $z = -2 + 3t$. [**Hint:** Consider the points $Q(1, 1, -2)$ and $R(2, 3, 1)$ which are on the line. Then a normal vector of $\mathbf{\Pi}$ is $\overrightarrow{PQ} \times \overrightarrow{PR}$.]

46. Find an equation of the plane $\mathbf{\Pi}$ passing throught the point $(6, 0, 2)$ and is perpendicular to the line $\boldsymbol{L}$ of intersection of the planes

$$\mathbf{\Pi}_1 : \ \ x + y + 2z = 4 \ \text{ and } \ \mathbf{\Pi}_2 : \ \ 3x - 2y + z = 1.$$

[**Hint:** Consider $\boldsymbol{N} = \boldsymbol{N}_1 \times \boldsymbol{N}_2 = (1, 1, 2) \times (3, -2, 1)$ as a normal vector to $\mathbf{\Pi}$.]

47. Let L_1 be the line of intersection of the planes:

$$x + 2y - z - 4 = 0 \text{ and } 2x + 5y - 3z - 9 = 0.$$

(a) Find parametric equations for L_1.

(b) Show that L_1 and $L_2 : x = 2 + 3t, y = 1 + 2t, z = 4 + t$, are perpendicular lines, but they do not intersect.

(c) Find an equation for the plane that contains L_1 and it is parallel to L_2.

48. Let L be the line through the points $P_1(-4, 2, -6)$ and $P_2(1, 4, 3)$.

(a) Find parametric equations for L.

(b) Find two planes whose intersection is L.

49. (a) Find parametric equations for the line L_1 which passes through the points $P(1, 2, 3)$ and $Q(4, 3, 2)$.

(b) Does L_1 intersect the line $L_2 : \frac{x-1}{2} = \frac{y-3}{1} = \frac{z-2}{3}$? If so, find their point of intersection.

50. (a) Find an equation of the plane Π through the points $P(2, 1, -2)$, $Q(-1, 1, 0)$, and $R(3, -1, -1)$.

(b) Find all values of k for which the point $(1, 1 - 2k, 3k - 1)$ lies on the plane Π.

51. Find the parametric equations for the line L which passes through the points $P(2, -1, 4)$ and $Q(4, 4, -2)$. For what value of k is the point $R(k + 2, 14, -14)$ on the line L?

52. Find the point of intersection of the line $x = 1 - t, y = 1 + t, z = t$, and the plane $3x + y + 3z - 1 = 0$.

53. Let $P(2, -2, 1)$, $Q(-1, 0, 3)$, $R(5, -3, 2)$, and $S(7, 17, -6)$ be points in $\mathbb{R}^3$.

(a) Find an equation of the plane Π through P, Q, and R and show that S does not lie on Π.

(b) Find the point T on the plane Π such that the vector $\overrightarrow{ST}$ is orthogonal to Π. [Hint: Find parametric equation for the line through S and perpendicular to Π.]

54. Determine if the following parametric equations represent the same line

$$L_1 : x = 4 - t, y = 2 + 3t, z = 1 + 2t, \quad \text{and} \quad L_2 : x = 1 + 3r, y = 11 - 9r, z = 7 - 6r.$$

55. Find the equations in symmetric form of line of intersection of planes:

$$\Pi_1 : x + 2y - z = 2, \quad \text{and} \quad \Pi_2 : 3x + 7y + z = 11.$$

56. Find an equation of the plane containing the lines

$$L_1 : x = 3 + t, y = 1 - t, z = 3t, \qquad \text{and} \qquad L_2 : x = 2s, y = -2 + s, z = 5 - s.$$

57. Let L be the line which is perpendicular to the plane $\Pi : 3x - y + 4z = 0$, and passes through the point $P(2, 6, 7)$. Find symmetric equations for L.

58. Let $\Pi_1 : 2x + 8y + 5z = 3$ and $\Pi_2 : x + 4y + 2z = 1$ be two planes, and let L be the line of intersection of Π_1 and Π_2. Find parametric equations for L.

59. Write an equation for the plane which contains the two lines:

$$L_1 : \frac{x-1}{1} = \frac{y+1}{2} = \frac{z}{-3} \qquad \text{and} \qquad L_2 : x = 1 - t, y = 3 - 2t, z = 2 + 3t.$$

60. Find $a, b \in \mathbb{R}^3$ so that the point $P(3, a - 2b, 2a + b)$ lies on the line

$$L : x = 1 + 2t, y = 2 - t, z = 4 + 3t.$$

61. Let $A(1, 1, 1)$, $B(1, 2, 2)$ and $C(2, 3, 4)$ be three points in $\mathbb{R}^3$.

   (a) Find the area of the triangle whose vertices are A, B and C.

   (b) Find parametric equations of the line L passing through the points B and C.

   (c) Find an equation of the plane Π containing the point A and orthogonal to $\overrightarrow{BC}$.

   (d) Find the point of intersection, if any, between the line L and the plane Π.

62. Answer each of the following as True or False:

   (a) (T) The plane $5y + 3z - 7 = 0$ is parallel to the vector $(1, 0, 0)$.

   (b) (T) In $\mathbb{R}^3$,if $X = 4Y$, then $X \times Y = (0, 0, 0)$.

   (c) (F) In $\mathbb{R}^3$, the plane $\pi : y = 3$ is parallel to the y-axis.

   (d) (T) The vector $X \times Y$ is perpendicular to $2X + 3Y$.