Exercises for Euclidean Plane Geometry: Math 226

Dr. Abdullah Al-Azemi

Mathematics Department
Kuwait University

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Chapter

1 Triangles and Quadrilaterals

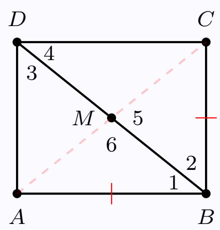
1.2 Exercises 1.2

Example 1.2.1

Let $ABCD$ be a quadrilateral such that $\overline{BD}$ bisects $\hat{B}$. Show that

1. if $\overline{AB} \cong \overline{BC}$, then $\overline{BD}$ also bisects $\hat{D}$.
2. if (1) is the case, show that $\overline{AC} \perp \overline{BD}$.

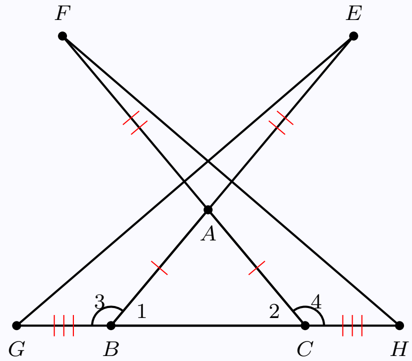
Solution:



Example 1.2.2

Given $\overline{AB} \cong \overline{AC}$; $\overline{AE} \cong \overline{AF}$ and $\overline{BG} \cong \overline{CH}$. Show that $\hat{E} \cong \hat{F}$.

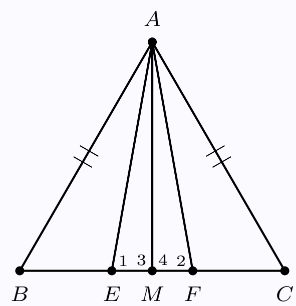
Solution:



Example 1.2.3

Let ABC be a triangle with $\overline{AB} \cong \overline{AC}$. Let E and F be two points on $\overline{BC}$ so that $\overline{AE}$ bisect $B\hat{A}F$, and $\overline{AF}$ bisect $E\hat{A}C$. Show that if M is the midpoint of $\overline{EF}$, then $\overline{AM}$ is perpendicular to $\overline{BC}$.

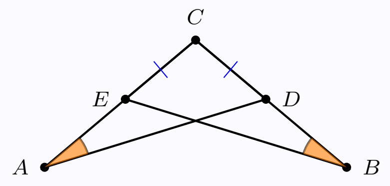
Solution:



Example 1.2.4

Given: $\overline{CD} \cong \overline{CE}$ and $\hat{A} \cong \hat{B}$. Prove that $\triangle\, CAD \cong \triangle\, CBE$.

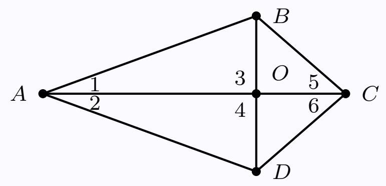
Solution:



Example 1.2.5

Given: $\hat{1} \cong \hat{2}$; and $\hat{5} \cong \hat{6}$. Prove that $\overline{AC} \perp \overline{BD}$.

Solution:



Example 1.2.6

Prove Theorem 1.1.2. Simply use the triangle congruence.

Example 1.2.7

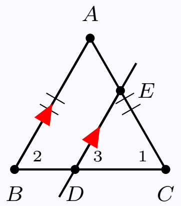
Prove Theorem 1.1.3. Simply use the triangle congruence.

1.3 Exercises 1.3

Example 1.3.1

Let ABC be an isosceles triangle, and let l be any line parallel to one of the sides and cutting the other two sides in two distinct points. Prove that the triangle formed by a vertex and these two points is also an isosceles.

Solution:



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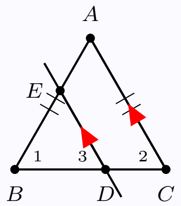

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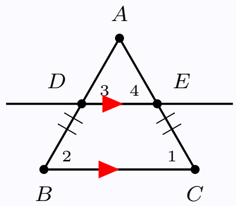
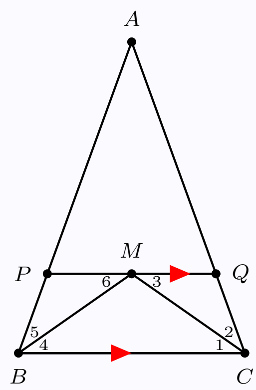


③

Example 1.3.2

Let ABC be a triangle such that the bisectors of $\hat{B}$ and $\hat{C}$ meets in M. If the line through M parallel to $\overline{BC}$ meets $\overline{AB}$ in P and $\overline{AC}$ in Q, then show that $\left|\overline{PQ}\right| = \left|\overline{PB}\right| + \left|\overline{QC}\right|$. Moreover, show that the $\triangle\, MBC$ is an obtuse triangle. Can you find $\left|B\hat{M}C\right|$ in terms of $\left|\hat{A}\right|$.

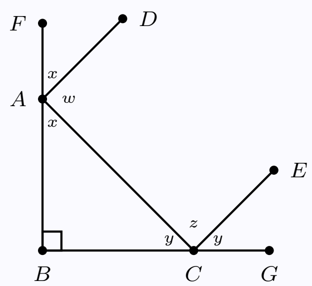
Solution:



Example 1.3.3

Given that $\left|\hat{B}\right| = 90°$; $\left|F\hat{A}D\right| = \left|C\hat{A}B\right|$; and $\left|G\hat{C}E\right| = \left|A\hat{C}B\right|$. Show that $\overline{AD} \parallel \overline{CE}$.

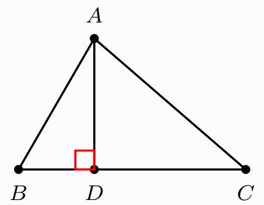
Solution:



Example 1.3.4

Show that: In any triangle $\triangle ABC$, we have $|\overline{BC}| < |\overline{AB}| + |\overline{AC}|$.

Solution:



Example 1.3.5

Show that: A triangle is equilateral if and only if it is equiangular.

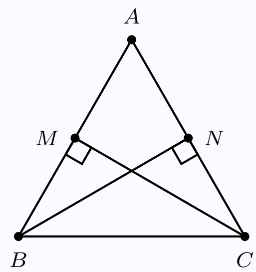
Solution:

1.4 Exercises 1.4

Example 1.4.1

Given a triangle ABC such that the altitudes $\overline{CM}$ and $\overline{BN}$ are congruent. Show that $\overline{AB} \cong \overline{AC}$ (Or show that $\triangle\ ABC$ is isosceles triangle).

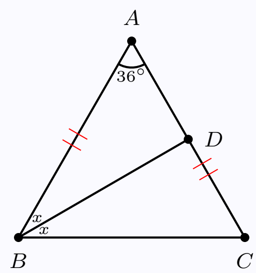
Solution:



Example 1.4.2

Given a triangle ABC such that $\cong ABAC$ and $\left|\hat{A}\right| = 36°$. If the bisector of $\hat{B}$ meets $\overline{AC}$ in D. Show that $\left|\overline{BC}\right|^2 = \left|\overline{AC}\right| \cdot \left|\overline{DC}\right|$.

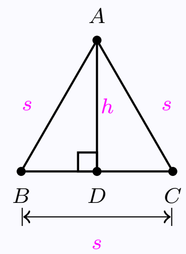
Solution:



Example 1.4.3

Find the measure of the sides of an equilateral triange ABC in terms of the measure of the altitude.

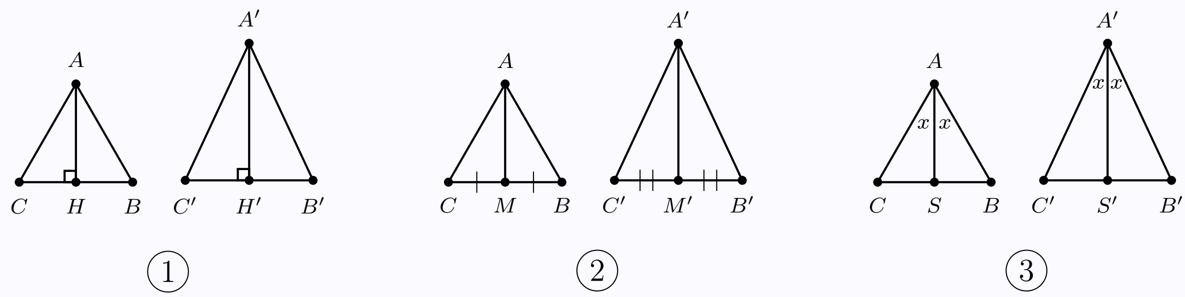
Solution:



Example 1.4.4

Given $\triangle ABC \sim \triangle A'B'C'$. Show that the ratio of measures of the corresponding (1): altitudes, (2): medians, and (3): angle bisectors is the same as the ratio of corresponding sides.

Solution:

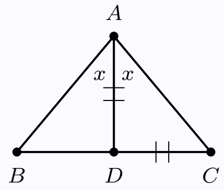


Example 1.4.5

In a triangle ABC, $\overline{AD}$ is angle bisector of $\hat{A}$ and $\overline{AD} \cong \overline{DC}$.

Show that $\left|\overline{AB}\right|^2 = \left|\overline{BD}\right| \cdot \left|\overline{BC}\right|$.

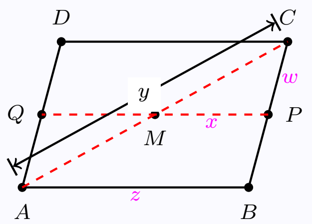
Solution:



Example 1.4.6

Let $ABCD$ be a parallelogram with Q in $\overline{AD}$ and P in $\overline{BC}$ so that $\overline{PQ} \parallel \overline{AB}$. Find $\left|MC\right|$ in terms of x, y, z, w.

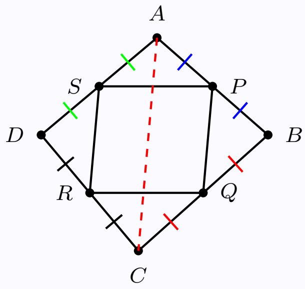
Solution:



Example 1.4.7

Let $ABCD$ be any quadrilateral, with P, Q, R, and S be the midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$. Show that $PQRS$ is a parallelogram.

Solution:

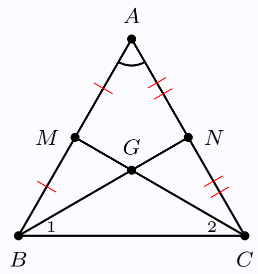


1.5 Exercises 1.5

Example 1.5.1

Let ABC be any triangle with CM and BN are the medians. Show that $\overline{CM} \cong \overline{BN}$ if and only if $\overline{AB} \cong \overline{AC}$.

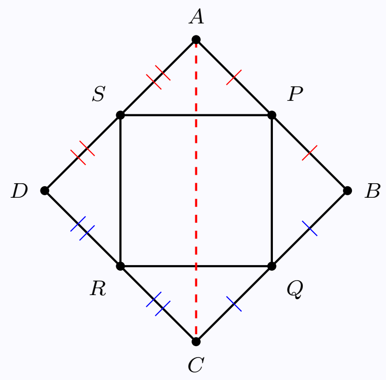
Solution:



Example 1.5.2

Let $ABCD$ be a quadrilateral with P, Q, R, and S the mid points of the sides as its shown. Show that $PQRS$ is a parallelogram.

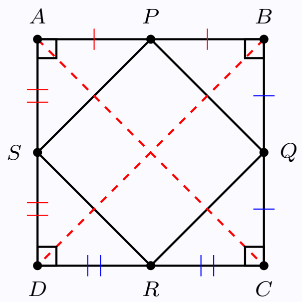
Solution:



Example 1.5.3

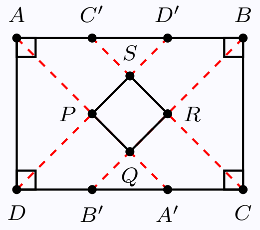
If $ABCD$ of Problem 1.5.2 is a rectangle with midpoints P, Q, R, and S of the sides, show that $PQRS$ is a rhombus. What is can be concluded about $PQRS$ if $ABCD$ was a square?

Solution:



Example 1.5.4

Let $ABCD$ be a rectangle. Show that the bisectors of its angles form a square.

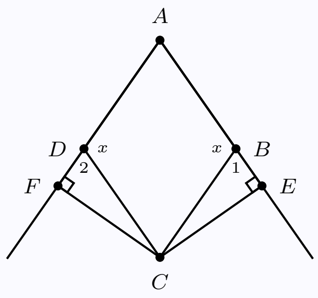


Solution:

Example 1.5.5

Given $ABCD$ is a rhombus. Show that C is equidistant from $\overline{AB}$ and $\overline{AD}$.

Solution:

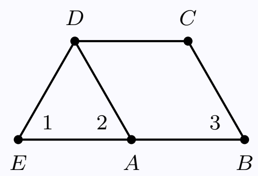


Example 1.5.6

Let $ABCD$ be a parallelogram: Show that

1. if $\overline{AB} \cong \overline{DE}$, and $\hat{1} \cong \hat{2}$, then $ABCD$ is a rhombus.
2. if $\overline{BC} \cong \overline{DE}$, then $\hat{1} \cong \hat{2}$ and $\hat{1} \cong \hat{3}$.

Solution:



Chapter

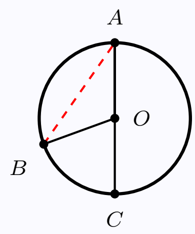
2 Circles

2.1 Exercises 2.1

Example 2.1.1

If $\left|\hat{A}\right| = n$. Show that $\left|\widehat{\mathrm{BC}}\right| = 2n$.

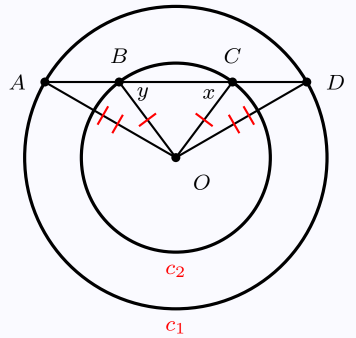
Solution:



Example 2.1.2

Let $c_1(O, r_1)$ and $c_2(O, r_2)$ be two circles with the same center O and $r_1 > r_2$. A chord $\overline{AD}$ in c_1 intersecting c_2 in B and C (drawn). Show that $\overline{AB} \cong \overline{CD}$. Can you show that $\overline{AC} \cong \overline{BD}$.

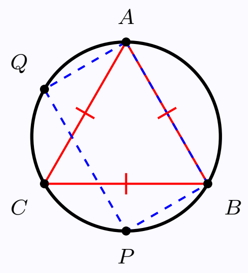
Solution:



2.2 Exercises 2.2

Example 2.2.1

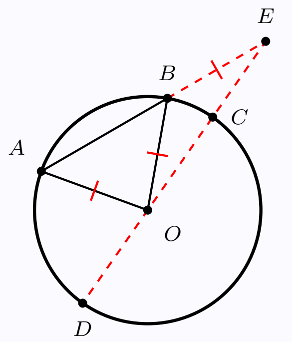
Equilateral $\triangle ABC$ is inscribed in a circle. Let P and Q be midpoints of $\overset{\frown}{\mathrm{BC}}$ and $\overset{\frown}{\mathrm{AC}}$, respectively. Show that $AQPB$ is a rectangle.



Solution:

Example 2.2.2

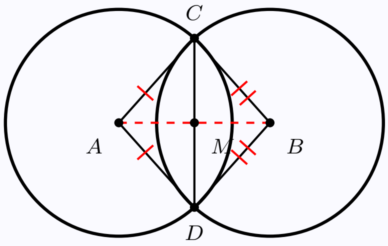
Let $\overline{AB}$ be a chord in a circle $c(O, r)$. $\overline{AB}$ is extended to a point E such that $\left|\overline{BE}\right| = r$. Join E with the center O with intersection at point D as drawn. Show that $\left|A\hat{O}D\right| = 3\left|D\hat{E}B\right|$.



Solution (1):

Example 2.2.3

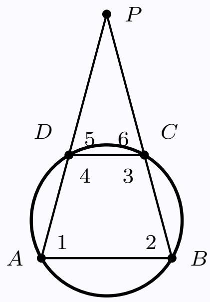
Let $\odot A$ and $\odot B$ be two circles intersecting in C and D. Show that $\overline{AB}$ is a perpendicular bisector of $\overline{CD}$.



Solution:

Example 2.2.4

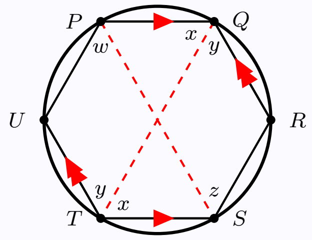
Let $ABCD$ be a cyclic quadrilateral. If $\overline{AD}$ meets $\overline{BC}$ in P, show that $\triangle PAB \sim \triangle PCD$.



Solution:

Example 2.2.5

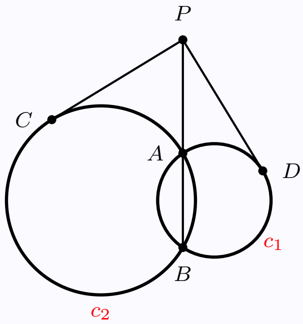
Let P, Q, R, S, T, U (in this order) be six points on a circle such that $\overline{PQ} \parallel \overline{TS}$ and $\overline{QR} \parallel \overline{UT}$. Show that $\overline{RS} \parallel \overline{PU}$.



Solution:

Example 2.2.6

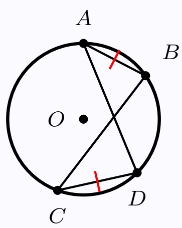
Let $c_1(O_1, r_1)$ and $c_2(O_2, r_2)$ be two circles intersecting in A and B. P is a point on the line $\overline{AB}$ outside both circles. $\overline{PC}$ and $\overline{PD}$ are tangents to c_1 and c_2, respectively. Show that $\overline{PC} \cong \overline{PD}$.



Solution:

Example 2.2.7

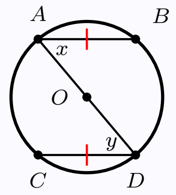
The chords $\overline{AB}$ and $\overline{CD}$ in the diagram are congruent. Show that $\overline{AD} \cong \overline{BC}$.



Solution:

Example 2.2.8

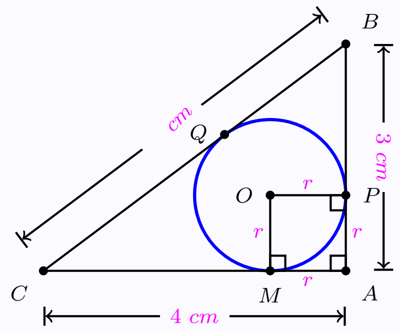
Two chords $\overline{AB}$ and $\overline{CD}$ are congruent. If $\overline{AD}$ passes through the center, show that $\overline{AB} \parallel \overline{CD}$.



Solution:

Example 2.2.9

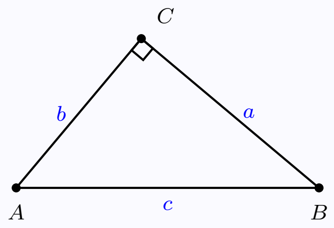
Given triangle ABC, right angled at A with $\left|\overline{AB}\right| = 3cm$ and $\left|\overline{AC}\right| = 4cm$. Find the radius of the inscribed circle.



Solution:

Example 2.2.10

Angle C of $\triangle ABC$ is a right angle. The sides of the triangle have the lengths shown. The smallest circle (not shown) through C that is tangent to $\overline{AB}$ intersects $\overline{AC}$ at J and $\overline{BC}$ at K. Express the distance JK in terms of a, b, and c.



Solution:

Chapter

3 Locus

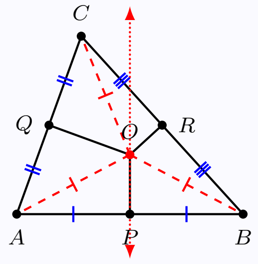
3.1 Exercises 3.1

Example 3.1.1

A ladder of length 4 m is stand against vertical wall. If the bottom of the ladder slides away and finally became horizontaly at the ground, find the locus of midpoints of the ladder.

Example 3.1.2

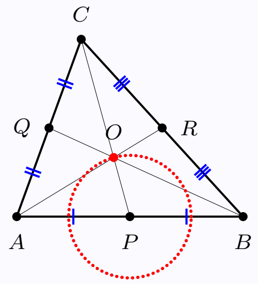
Given a triangle ABC with A and B are fixed and C moves in the plane. Find the locus of the circumcenter of $\triangle\ ABC$.



Solution:

Example 3.1.3

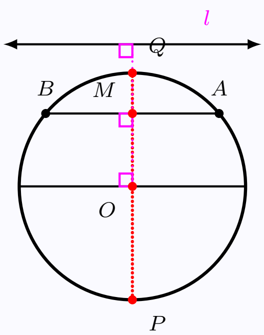
Given a triangle ABC with A and B are fixed and C moves in the plane such that the median $\overline{CP}$ has a constant measure. Find the locus of the centroid of $\triangle\ ABC$.



Solution:

Example 3.1.4

Given a circle $c(O, r)$ and a line $\overleftrightarrow{l}$ in the plane. A chord $\overline{AB}$ moves in the circle such that $\overline{AB} \parallel \bar{l}$. Find the locus of the midpoints of $\overline{AB}$.



Solution:

Chapter

4 Transformations

4.2 Exercises 4.2

Example 4.2.1

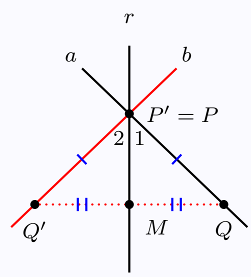
Let $\mathbf{R}_r$ be the reflection in line r. If a is a line with $\mathbf{R}_r(a) = b$, show that a is parallel to r iff b is parallel to r.

Solution:

Example 4.2.2

Let $\mathbf{R}_r$ be the reflection in line r. If a is a line with $\mathbf{R}_r(a) = b$, show that r is the locus of points that are equidistant from lines a and b.

Solution:

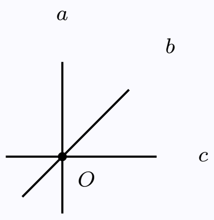


Case 2

Example 4.2.3

Let a, b, c be three lines concurrent in O. What type of isometry is $\mathbf{R}_c\mathbf{R}_b\mathbf{R}_a$?

Explain your answer.



Solution:

4.3 Exercises 4.3

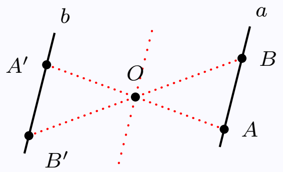
Example 4.3.1

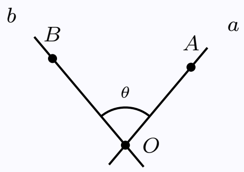
Let a be any line in the plane. Find all rotation that leaves a invariant.

Solution:

Example 4.3.2

Let a, b be any two line in the plane. Is there a rotation $\mathcal{R}_{O,\theta}$ that takes a to b?

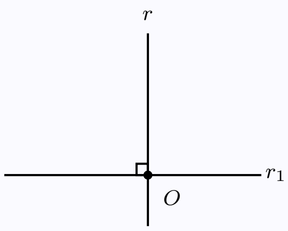




Solution:

Example 4.3.3

Let $\mathbf{R}_r$ be a reflection in line r. If O is a point on r, and $\mathcal{H}_O$ is the half-turn about O, then find $\mathbf{R}_r\mathcal{H}_O$ and $\mathcal{H}_O\mathbf{R}_r$.



Solution:

Example 4.3.4

For an arbitrary triangle $\triangle\, ABC$ as in the diagram what can be said, in general, about the isometry $\mathcal{R}_{A,\alpha} \circ \mathcal{R}_{B,\beta} \circ \mathcal{R}_{C,\gamma}$, where α, β, γ are, respectively, the vertex angles at A, B, C, respectively?

4.4 Exercises 4.4

Example 4.4.1

What is the product of a translation and a half-turn.

Solution:

Example 4.4.2

Given a reflection $\mathbf{R}_c$ for some line c, and a translation $\mathcal{T}_{\overrightarrow{PQ}}$ in the same direction of c, show that $\mathbf{R}_c \circ \mathcal{T}_{\overrightarrow{PQ}} = \mathcal{T}_{\overrightarrow{PQ}} \circ \mathbf{R}_c$.

Solution:

Example 4.4.3

Given a translation $\mathcal{T}_{\rightarrow} = \mathbf{R}_a\mathbf{R}_b$. Show that $\mathcal{T}_{\rightarrow}^{-1} = \mathbf{R}_b\mathbf{R}_a$.

Solution:

Chapter

5 Homothecy and Similarity

5.1 Exercises 5.1

Example 5.1.1

Let $\overline{AM}, \overline{CN}$ be two medians of a triangle $\triangle ABC$. Extend $\overline{AM}$ and $\overline{AN}$ to two points M' and N', respectively, so that $\left|\overline{AM}\right| = \left|\overline{MM'}\right|$ and $\left|\overline{CN}\right| = \left|\overline{NN'}\right|$. Show that M', B, N' are collinear.

Solution:

Example 5.1.2

Let $\triangle\ ABC$ be a given triangle and let $\lambda, \mu, \gamma > 0$. Show that

$$\mathcal{D}_{A,\lambda}\left(\triangle\ ABC\right) \sim \mathcal{D}_{B,\mu}\left(\triangle\ ABC\right) \sim \mathcal{D}_{C,\gamma}\left(\triangle\ ABC\right).$$

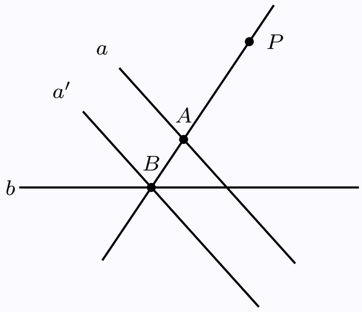
Moreover, if $\lambda = \mu$, then $\mathcal{D}_{A,\lambda}\left(\triangle\ ABC\right) \cong \mathcal{D}_{B,\beta}\left(\triangle\ ABC\right)$.

Solution:

5.2 Exercises 5.2

Example 5.2.1

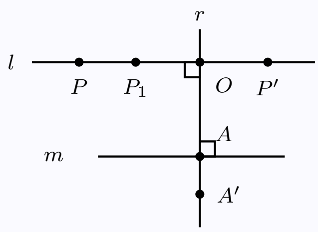
Given two intersecting lines a and b in the plane and a point P not on any of them. If $k > 0$ is a given real number, then show that there exists a unique line through P that cuts a and b in A and B, respectively, such that $\dfrac{|\overline{PB}|}{|\overline{PA}|} = k$.



Solution:

Example 5.2.2

Let $\mathbf{T}$ be a dilative reflection that is not an isometry. If $\mathbf{T} = \mathbf{R}_r\mathcal{D}_{O,\lambda}$, then find all invariant lines under $\mathbf{T}$.



Solution:

Chapter

6 Coordinate Geometry

6.1 Exercises 6.1

Example 6.1.1

Let $a: \ 2x + y = 5$ and $b: \ 6x + 3y = 11$ be two lines. If $A(0, 5)$ and $B(1, 3)$, then:

1. Show that A and B are points of a.
2. Show that a and b are parallel.
3. Calculate $d(A, b)$ and $d(B, b)$. What do you notice?

Solution:

Example 6.1.2

Let $l_1 : 2x + y - 5 = 0$ and $l_2 : x - 3y - 1 = 0$ be two lines. Find the locus of points equidistants from l_1 and l_2.

Solution:

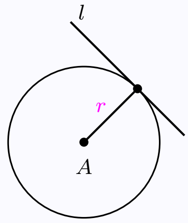
Example 6.1.3

Find the locus of points equidistant from $A(1, 5)$ and $B(-3, 7)$.

Solution:

Example 6.1.4

Find an equation of the circle with center $A(1, 5)$ and tangent to the line $l: \ x + y = 2$.



Solution:

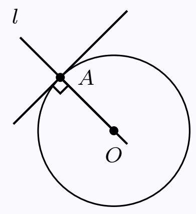
Example 6.1.5

Find an equation of the line l that passes through the point $A(3, -5)$ and the center of the circle: $x^2 + y^2 + 2x - 2y + 1 = 0$.

Solution:

Example 6.1.6

Find an equation of the line l tangent to the circle $x^2 + y^2 = 25$ at the point $A(-4, 3)$.



Solution:

Example 6.1.7

Find the locus of points equidistant from the lines $l_1 : \ x - y + 1 = 0$ and $l_2 : \ x + 7y - 49 = 0$.

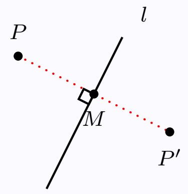
Solution:

6.2 Exercises 6.2

Example 6.2.1

Reflect the point $P(3, 1)$ in the line $l: \; y - 2x = 0$.

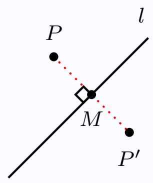
Solution:



Example 6.2.2

Find $\mathbf{R}_l\,(P)$, where $l: \; y - x + 1 = 0$ and $P(2, 3)$.

Solution:



Example 6.2.3

Find $\mathbf{R}_l\,(P)$, where $l:\ x+y-9=0$ and $P(2,5)$.

Solution:

Example 6.2.4

If $P'(4,6)$ is the reflection of the point $P(0,2)$ in line l, then find an equation of l.

Solution:

Example 6.2.5

The line $a:\ y = 2x + 3$ is reflected in the y-axis. Find an equation of the image line b.

Solution:

Example 6.2.6

Let $A(-4, 2)$ and $B(2, 6)$, and let $\mathbf{R}_l$ be a reflection that maps A to B. Find an equation of l.

Solution:

Example 6.2.7

Show that the product of translations is commutative.

Solution:

Example 6.2.8

Show that translations are isometries.

Solution:

Example 6.2.9

Find the image of the circle $(x-1)^2+(y-2)^2=6$ under each of the following:

1. A half-turn in $\mathcal{H}_{(-3,4)}$.
2. A translation: $\mathcal{T}_{5,-7}$.
3. A reflection in the line $x=-1$.
4. A reflection in the line $y=3$.
5. A homethecy $\mathcal{D}_{O,-2}$.

Solution: