1. **(3+3+2 pts.)**

(a) Suppose that $\triangle AMC \cong \triangle BMD$. Show that $\triangle ABC \cong \triangle BAD$.

(b) Use Similarity-Angle Angle (S-AA) to show that $\triangle AMB \sim \triangle DMC$.

(c) Let $PQRS$ be a parallelogram. If $\overline{QR} \cong \overline{ST}$ and $|\hat{1}| = 45°$, then find $|\hat{2}|$ and $|\hat{3}|$.

<u>**Solution**</u>:

Figure: Part (a) and (b)

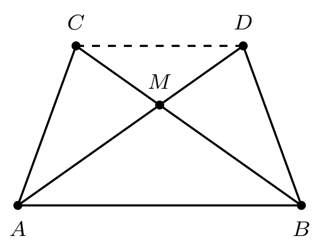
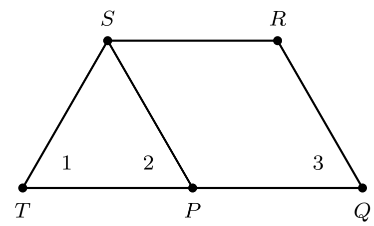


Figure: Part (c)



(a) Note that $\overline{CM} \cong \overline{DM}$ and $\overline{AM} \cong \overline{BM}$ and hence $\overline{AD} \cong \overline{BC}$. Also, we have $\overline{AC} \cong \overline{BD}$. Furthermore, $\overline{AB}$ is common in $\triangle ABC$ and $\triangle BAD$. Therefore, by SSS, $\triangle ABC \cong \triangle BAD$.

(b) Let $\left|C\hat{M}D\right| = \left|A\hat{M}B\right| = x$. Since $\triangle CMD$ and $\triangle AMB$ are both isosceles, we have $\left|M\hat{A}B\right| = \left|M\hat{B}A\right| = 180 - x$. Hence $\left|M\hat{A}B\right| = \left|M\hat{B}A\right| = 90 - \frac{x}{2}$. In a similar way, we can show that $\left|M\hat{C}D\right| = \left|M\hat{D}C\right| = 90 - \frac{x}{2}$. Hence by S-AA, we have $\triangle AMB \sim \triangle DMC$.

(c) Clearly, $\overline{PS} \cong \overline{QR} \cong \overline{ST}$. Hence, $\triangle PST$ is isosceles triangle and hence $|\hat{2}| = 45°$. Moreover, $\hat{2} \cong \hat{3}$ since they are corresponding angles. Thus $|\hat{3}| = 45°$ as well.

2. (**4 pts. each**)

(a) The chords $\overline{AB}$ and $\overline{CD}$ in the diagram are congruent. Show that $\overline{AD} \cong \overline{BC}$.

(b) In the diagram, let $\overline{AB}$ be a diameter in the circle, $\left|\widehat{\mathrm{BC}}\right| = 40°$, and $\left|\hat{1}\right| = 60°$. Find $\left|\widehat{\mathrm{BD}}\right|$.

<u>Solution</u>:

Figure: Part (a)

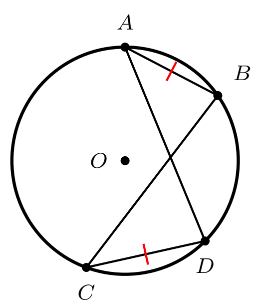
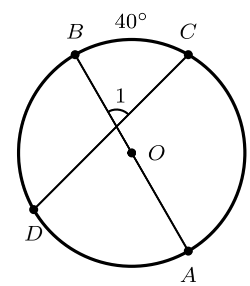


Figure: Part (b)



(a) Since $\overline{AB} \cong \overline{CD}$, we have $\widehat{\mathrm{AB}} \cong \widehat{\mathrm{CD}}$. Then $\left|\widehat{\mathrm{AB}}\right| + \left|\widehat{\mathrm{BD}}\right| = \left|\widehat{\mathrm{CD}}\right| + \left|\widehat{\mathrm{BD}}\right|$. That is $\left|\widehat{\mathrm{AD}}\right| = \left|\widehat{\mathrm{BC}}\right|$. Therefore, $\overline{AD} \cong \overline{BC}$.

(b) Clearly, by Theorem in class $\left|\hat{1}\right| = \frac{1}{2}\left(\left|\widehat{\mathrm{BC}}\right| + \left|\widehat{\mathrm{AD}}\right|\right)$. That is $60° = \frac{1}{2}(40° + \left|\widehat{\mathrm{AD}}\right|)$. Hence $\left|\widehat{\mathrm{AD}}\right| = 120° - 40° = 80°$. Therefore, $\left|\widehat{\mathrm{BD}}\right| = 100°$

3. (**3+3+2 pts.**)

(a) The rotation $\mathcal{R}_{O,x}$ maps line a to line b. In the diagram, what is the measure of the angle from a to b? Show your work.

(b) $\triangle\, ABC$ and $\triangle\, DEC$ are isosceles right triangles. Show that $\overline{AD} \perp \overline{BE}$.

(c) Let $\mathbf{T}$ be a dilative reflection that is not an isometry, and let O be a point on the line r. If $\mathbf{T} = \mathbf{R}_r \mathcal{D}_{O,\lambda}$ "reflection after dilation", then r is invariant under $\mathbf{T}$. Find another invariant line under $\mathbf{T}$.

<u>**Solution**</u>:

Figure: Part (a)

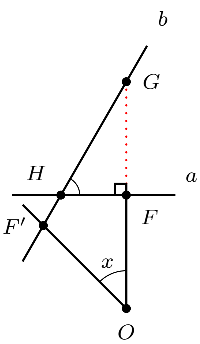


Figure: Part (b)

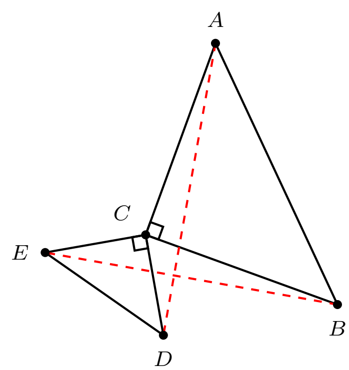
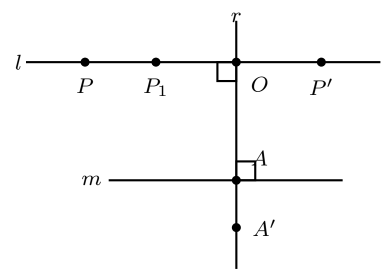


Figure: Part (c)



(a) Let F be a point on a so that $\overline{OF} \perp a$. Thus, $\mathcal{R}_{O,x}(F) = F' \in b$.
Let G be the point of intersection of $\overline{OF}$ with b, and let H be the intersection point of a with b. Since $\left|O\hat{F}H\right| = 90$, we get $\left|O\hat{F'}H\right| = 90$ as well since rotation preserves angles. Thus, we get $\left|F'\hat{G}O\right| = 90 - x$. Therefore, $\left|G\hat{H}F\right| = 90 - (90 - x) = x$.

(b) This is an application of part (a): Note that $\mathcal{R}_{C,90}(B) = A$ and $\mathcal{R}_{C,90}(E) = D$. Hence, under the same rotation, we have $\overline{BE}$ maps to $\overline{AD}$. That is $\mathcal{R}_{C,90}(\overline{BE}) = \overline{AD}$. Therefore, $\overline{BE} \perp \overline{AD}$ since the angle inbetween is $90°$.

(c) $\mathbf{T}$ is not isometry and hence $\lambda \neq \pm 1$ and $\mathbf{T} \neq \mathbf{I}$.
Line: Let l be perpendicular to r at O. Then $\mathcal{D}_{O,\lambda}(P) = P_1$ and $\mathbf{R}_r(P_1) = P' \in l$. Therefore, l is invariant. There are no other invariant lines under $\mathbf{T}$ since any line m must be perpendicular to r to be invariant under reflection. But if it does not contain O, then for any point $A \in m$, $\mathcal{D}_{O,\lambda}(A) = A' \notin m$. It is not invariant.

4. (3 pts. each)

(a) Let $l_1 :\ 2x + y - 5 = 0$ and $l_2 :\ x - 3y - 1 = 0$ be two lines. Find the locus of points equidistants from l_1 and l_2.

(b) Find an equation of the line l tangent to the circle $x^2 + y^2 = 25$ at the point $A(-4, 3)$.

(c) Find an equation of the circle with center $A(-3, 2)$ and tangent to the y-axis. Furthermore, find the points of intersection of the circle and x-axis.

Solution:

(a) Notice that if $l_1 \parallel l_2$, then the locus is a line that is parallel to both lines l_1 and l_2. Otherwise, the locus is two lines which are angle bisectors of the two lines.

Let $M(x, y)$ be the points of the locus. Thus,

$$d(M, l_1) = d(M, l_2) \ \Rightarrow\ \frac{|\,2x + y - 5\,|}{\sqrt{4+1}} = \frac{|\,x - 3y - 1\,|}{\sqrt{1+9}} \ \Rightarrow \sqrt{2}\,|\,2x + y - 5\,| = |\,x - 3y - 1\,|\,.$$

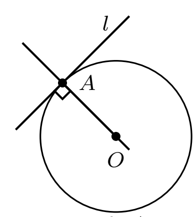
That is we have two cases:

Case 1: $\sqrt{2}(\,2x + y - 5\,) = +(\,x - 3y - 1\,)$, and hence $(2\sqrt{2} - 1)x + (\sqrt{2} + 3)y - 5\sqrt{2} + 1 = 0$.

Case 2: $\sqrt{2}(\,2x + y - 5\,) = -(\,x - 3y - 1\,)$, and hence $(2\sqrt{2} + 1)x + (\sqrt{2} - 3)y - 5\sqrt{2} - 1 = 0$.

Therefore, the locus is formed by the previous two lines. We obtain here that l_1 and l_2 are not parallel.

(b) The center of the circle is the origin $O(0, 0)$ and its radius is 5. Thus, the slope of $\overline{OA}$ is $\frac{3-0}{-4-0} = -\frac{3}{4}$. Therefore, $m_l = \frac{4}{3}$ since $l \perp \overline{OA}$. Therefore, $l :\ (y - 3) = \frac{4}{3}(x + 4)$.



(c) Since the circle is tangent to y-axis, we have the radius equals to the x-coordinates of A which is the distance from y-axis to A. Thus, $r = |3| = 3$, and hence the circle equation: $(x + 3)^2 + (y - 2)^2 = 9$.

Substitute $y = 0$ in the circle equation to get $x = \pm\sqrt{5} - 3$. Thus the points of intersection are $(\sqrt{5} - 3, 0)$ and $(-\sqrt{5} - 3, 0)$.

5. (**3 pts. each**) Find the image of the circle $(x-1)^2+(y-2)^2=1$ under each of the following:

(a) A reflection in the line $y-2x=0$.

(b) A rotation $\mathcal{R}_{(1,0),\frac{\pi}{2}}$.

(c) A dilation $\mathcal{D}_{O,-2}$.

<u>**Solution**</u>:

(a) Note that the center of the circle $P(1,2)$ lies on the line of reflection. Therefore, the circle is invariant, and hence the image is again: $(x-1)^2+(y-2)^2=1$.

(b) The rotation matrix about the origin through the angle $\frac{\pi}{2}$ is given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. But we are rotating about $(1,0)$. Thus the image of $P(1,2)$ is $P'(x,y)$ where

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Therefore, the image of the circle is: $(x+1)^2+y^2=1$.

(c) Dilation is **not** an isometry. Hence the new radius $r' = |\,\lambda\,|\,1 = |\,-2\,| = 2$.
Also, $\mathcal{D}_{O,-2}\,(A(1,2)) = A'(-2,-4)$. Thus,

$$c'(A',r')\,:\;\;(x+2)^2+(y+4)^2=4.$$