



Kuwait University
Faculty of Science
Department of Mathematics

Abstract Algebra 0410-262

Final Exam

Thursday, December 20, 2018
Fall 2018/19

Student Name									إسم الطالب
Student ID Number									الرقم الجامعي للطالب
									الرقم التسلسلي Serial Number

Section No. رقم الشعبة	Instructor Name أستاذ المقرر
01 A	Dr. Abdullah Alazemi

Instructions to students

Time allowed: 2 hours.

This exam contains 5 questions.

تعليمات للطالب

وقت الإختبار: ساعتين.

يحتوي هذا الإختبار على 5 أسئلة.

منوع دخول الآلات الحاسبة أو أي وسيلة للإتصال داخل قاعة الإختبار.

Calculators and communication devices are not allowed in the examination room.

Question 1	9
Question 2	9
Question 3	9
Question 4	6
Question 5	10
Total	43

Good Luck

1. (3 pts. each)

- (a) Show that if A is an abelian group with addition as the operation, and an operation $*$ is defined on A by $a * b = 0$ for all $a, b \in A$, then $(A, +, *)$ is a commutative ring.
- (b) Show that the integral domain $\mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a field.
- (c) Prove that if R and S are isomorphic rings and R is commutative, then S is commutative as well.

2. (3+2+2+2 pts.)

- (a) [**The Remainder Theorem**]: Show that if $f(x) \in F[x]$ and $c \in F$ (F is a field), then the remainder in the division of $f(x)$ by $x - c$ is $f(c)$.
- (b) Use the remainder theorem to compute the remainder when dividing $f(x) = x^3 - x^2 + x + 3$ by $x - 4$ in $\mathbb{Z}_5[x]$.
- (c) Is -1 a root for $f(x) = x^3 - x^2 + x + 3$.
- (d) $f(x) = x^3 - x^2 + x + 3$ irreducible over \mathbb{Z}_5 ? Why?

3. (3 pts. each)

- (a) Express $f(x) = x^3 + x$ as a product of irreducible polynomials in $\mathbb{Z}_5[x]$.
- (b) Let $\theta : R \rightarrow S$ be a ring homomorphism. Assuming that $\ker \theta$ is a subring of R , show that $\ker \theta$ is an ideal of R .
- (c) Let θ be a homomorphism of a ring R with unity e onto a nonzero ring S . Let u be a unit in R . Show that $\theta(u)$ is a unit in S .

4. (3 pts. each) Let I and J be two ideals of a ring R , and let $I + J$ be the subring of R defined by $I + J = \{a + b : a \in I \text{ and } b \in J\}$.

(a) Show that $I + J$ is an ideal of R containing each of I and J .

(b) Show that if $J \subseteq I$, then $I/J = \{i + J : i \in I\}$ is an ideal of R/J .

5. (3+1+3+3 pts. each)

- (a) Find all $c \in \mathbb{Z}_3$ such that $\mathbb{Z}_3[x] / (x^2 + c)$ is a field.
- (b) Find all $c \in \mathbb{Z}_3$ such that $(x^2 + c)$ is a maximal ideal of $\mathbb{Z}_3[x]$.
- (c) Let $\alpha \in \mathbb{Q}[x] / (x^2 - 7)$ be a root of the irreducible polynomial $x^2 - 7 \in \mathbb{Q}[x]$. Express $(1 + \alpha)^2$ in the form $a + b\alpha$ with $a, b \in \mathbb{Q}$.
- (d) Let $\mathbb{Z}_2(\alpha) = \{0, 1, \alpha, \alpha + 1\}$ be a field isomorphic to the field $\mathbb{Z}_2[x] / (x^2 + x + 1)$, where α is a root of $x^2 + x + 1$. Construct the multiplication table of $\mathbb{Z}_2(\alpha)$.