Kuwait University
Final Exam Revision
Math 363 - Advanced Linear Algebra

Mathematics Department
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Give full reasons for your answer. State clearly any Theorem you use.

1. Let $(\mathbb{V}, +, \cdot)$ be any vector space. Show that the zero vector in $\mathbb{V}$ is unique.

2. Let $\mathbb{W}_1$ be the set of all symmetric matrices and $\mathbb{W}_2$ be the set of all nonsingular matrices in $M_{n\times n}(\mathbb{R})$. Decide whether $\mathbb{W}_1$ and $\mathbb{W}_2$ are subspaces of $M_{n\times n}(\mathbb{R})$.

3. Show that $\mathbb{W} = \{\, A \in M_{n\times n}(\mathbb{R}) : tr(A) = 0 \,\}$ is a subspace of $M_{n\times n}(\mathbb{R})$.

4. Show that $\mathbb{W} = \left\{ \begin{pmatrix} a & a-b \\ a+b & b \end{pmatrix} : a, b \in \mathbb{R} \right\}$ is a subspace of $M_{2\times 2}(\mathbb{R})$.

5. Show that the set $S = \{\, 1-x, x-x^2, 1+x^2 \,\}$ spans $\mathbb{P}_2(\mathbb{R})$.

6. Show that the set $S = \{\, 1-x, x-x^2, 1+x^2 \,\}$ is linearly independent in $\mathbb{P}_2(\mathbb{R})$.

7. Let x and y be two linearly independent vectors in a vector space $\mathbb{V}$. Show that the condition for the vectors $ax + by$ and $cx + dy$ to be linearly dependent is $ad - bc = 0$.

8. Let $\mathbb{W} = \{\, (\, x, y, z, w\,) : x + y + z = 0 \text{ and } w = 2x \,\}$ be a subspace for $\mathbb{R}^4$. Find a basis for $\mathbb{W}$.

9. Let $\mathbb{W} = \{\, f(x) : f(1) = 0 \,\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{P}_2(\mathbb{R})$, and find its dimension.

10. Let $\mathbb{W} = \{\, a + bx + cx^2 : a = b = c \,\}$. Show that $\mathbb{W}$ is a subspace of $\mathbb{P}_2(\mathbb{R})$.

11. Let x and y be two distinct vectors of a vector space $\mathbb{V}$. Show that if $\beta = \{\, x, y \,\}$ is a basis for $\mathbb{V}$, then $\gamma_1 = \{\, x + y, ax \,\}$ and $\gamma_2 = \{\, ax, by \,\}$ are also bases for $\mathbb{V}$ for any nonzero scalars a and b.

12. Let $\mathbb{V}$ and $\mathbb{W}$ be two vector spaces, and let $\mathbf{T} : \mathbb{V} \to \mathbb{W}$ be linear. Show that $\mathbf{T}(0_{\mathbb{V}}) = 0_{\mathbb{W}}$.

13. Define $\mathbf{T} : M_{m\times n}(\mathbb{R}) \to M_{n\times m}(\mathbb{R})$ by $\mathbf{T}(A) = A^t$. Show that $\mathbf{T}$ is linear.

14. Let $\mathbf{T} : \mathbb{P}_1(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be a linear for which $\mathbf{T}(x-1) = x^2 + 1$ and $\mathbf{T}(x+1) = x(x-1)$. What is $\mathbf{T}(3x-1)$? Show your work.

15. Let $\mathbf{T}$ be the linear operator on $\mathbb{R}^2$ defined by $\mathbf{T}(x, y) = (\, 2x - 3y, y \,)$. Show that $\mathbf{T}$ is a bijection.

16. Let $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear defined by $\mathbf{T}(x, y) = (\, x + y, x - y, x \,)$. Find $\mathcal{N}(\mathbf{T})$, $\mathcal{R}(\mathbf{T})$, and find their bases. Is $\mathbf{T}$ one-to-one? Is $\mathbf{T}$ onto $\mathbb{R}^3$? Show your work.

17. Let $\mathbf{T}$ be the linear operator on $\mathbb{R}^2$ defined by $\mathbf{T}(x,y) = (2x+y, x-y)$. Find $\mathcal{N}(\mathbf{T})$, $\mathcal{R}(\mathbf{T})$, nullity($\mathbf{T}$), and rank($\mathbf{T}$).

18. Let $\mathbf{T} : \mathbb{P}_1(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R})$ be the linear defined by $\mathbf{T}(f(x)) = x \cdot f(x)$. (a) Find the matrix representation A for $\mathbf{T}$. (b) If $g(x) = 3x-2$, compute $[g(x)]_\gamma$, where γ is the standard ordered basis for $\mathbb{P}_2(\mathbb{R})$. (c) Evaluate $\mathbf{T}(g(x))$ using A.

19. Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$. Assume that $\mathbf{T} : \mathbb{P}_2(\mathbb{R}) \to \mathbb{P}_1(\mathbb{R})$ is the linear defined by A using the standard ordered basis β and γ for $\mathbb{P}_2(\mathbb{R})$ and $\mathbb{P}_1(\mathbb{R})$, respectively. Evaluate $\mathbf{T}(g(x))$, where $g(x) = 2x^2 - 3x + 1$.

20. Let $\beta = \{ (1,1), (1,-1) \}$ and $\gamma = \{ (2,4), (3,1) \}$ be two bases for $\mathbb{R}^2$. (1) Find the matrix Q that changes γ-coordinates into β-coordinates. (2) Use Q to evaluate $[(1,7)]_\beta$. (3) If $\mathbf{T}$ is the linear opertor on $\mathbb{R}^2$ defined by $\mathbf{T}(x,y) = (3x-y, x+3y)$, find $[\mathbf{T}]_\gamma$.

21. Let $\mathbf{T}$ be the linear operator on $\mathbb{P}_2(\mathbb{R})$ defined by $\mathbf{T}(f(x)) = f(x) + (x+1)\, f'(x)$. Find an ordered basis γ for $\mathbb{P}_2(\mathbb{R})$ so that $[\mathbf{T}]_\gamma$ is a diagonal matrix.

22. Let $\mathbf{T}$ be the linear operator on $\mathbb{P}_2(\mathbb{R})$ defined by $\mathbf{T}(f(x)) = f'(x)$. Is $\mathbf{T}$ diagonalizable? Explain.

23. Let $\mathbf{T}$ be the linear operator on $\mathbb{P}_2(\mathbb{R})$ defined by $\mathbf{T}(f(x)) = f(1) + f'(0)\, x + (f'(0) + f''(0))\, x^2$. Is $\mathbf{T}$ diagonalizable? Explain.

24. Let $\mathbb{V}$ be an inner product space . Show that $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for any $x, y, z \in \mathbb{V}$.

25. Let $\mathbb{V}$ be an inner product space , and let $y, z \in \mathbb{V}$. Show that if $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in \mathbb{V}$, then $y = z$.

26. Let $\mathbb{V} = \mathbb{C}^3$ with the standard inner product. If $x = (2, 1+i, i), y = (2-i, 2, 1+2i) \in \mathbb{V}$, then verify both the Cauchy-Schwarz inequality and the triangle inequality for x and y.

27. Let $\mathbb{V}$ be an inner product space , and let $S = \{ x_1, x_2, \cdots, x_n \}$ be an orthonormal basis for $\mathbb{V}$. Show that $y = \sum_{i=1}^{n} \langle y, x_i \rangle\, x_i$, for any $y \in \mathbb{V}$.

28. Let $S = \left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$ be an orthonormal basis for $\mathbb{R}^3$. Express (do not compute the coefficients) $x = (1,2,3)$ as a linear combination of vectors of S.

29. Let $\mathbb{W} = \{ (x+y, x, x+2y) : x, y \in \mathbb{R} \}$ be a subspace of $\mathbb{R}^3$. Find an orthonormal basis for $\mathbb{W}$.

30. Let $\mathbf{T}$ be the linear operator on $\mathbb{C}^2$ defined by $\mathbf{T}(x,y) = (2xi + 3y, x-y)$. Evaluate $\mathbf{T}^*$.

31. Let $\mathbf{T}$ be the linear operator on $\mathbb{R}^2$ defined by $\mathbf{T}(x, y) = (2x + y, x - 3y)$. Evaluate $\mathbf{T}(2, 3)$.

32. Let $\mathbf{T}$ be the linear operator on $\mathbb{P}_1(\mathbb{R})$ defined by $\mathbf{T}(f) = f' + 3f$ with $\langle f, g \rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$. Evaluate $\mathbf{T}^*(4 - 2x)$.

33. Let $\mathbb{V}$ be an inner product space , and let $y, z \in \mathbb{V}$. Define $\mathbf{T} : \mathbb{V} \to \mathbb{V}$ by $\mathbf{T}(x) = \langle x, y \rangle z$ for all $x \in \mathbb{V}$. Show that $\mathbf{T}$ is linear, and evaluate $\mathbf{T}^*(x)$.

34. Let $\mathbb{V}$ be an inner product space , and let $\mathbf{T}$ be a normal operator on $\mathbb{V}$. Show that

    (a) $\|\mathbf{T}(x)\| = \|\mathbf{T}^*(x)\|$ for all $x \in \mathbb{V}$.

    (b) If x is an eigenvector of $\mathbf{T}$, then x is an eigenvector of $\mathbf{T}^*$. In fact, if $\mathbf{T}(x) = \lambda x$, then $\mathbf{T}^*(x) = \overline{\lambda} x$.

    (c) If λ_1 and λ_2 are two distinct eigenvalues for $\mathbf{T}$ with corresponding eigenvectors x_1 and x_2, respectively, then x_1 and x_2 are orthogonal.

35. Let $\mathbf{T}$ be a linear operator on a finite-dimensional inner product space $\mathbb{V}$ with $\langle \mathbf{T}(x), \mathbf{T}(y) \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{V}$. Show that if β is an orthonormal basis for $\mathbb{V}$, then so is $\mathbf{T}(\beta)$.

36. Let $\mathbf{T}$ be a linear operator on an inner product space $\mathbb{V}$. If $U = \mathbf{T} + \mathbf{T}^*$, then show that U is normal.

37. Let $\mathbf{T}$ be the linear operator on $\mathbb{V} = \mathbb{R}^2$ defined by $\mathbf{T}(x, y) = (2x - 2y, -2x + 5y)$. Determine whether $\mathbf{T}$ is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of $\mathbf{T}$ for $\mathbb{V}$.

38. Let $\mathbf{T}$ be the linear operator on $\mathbb{V} = \mathbb{C}^2$ defined by $\mathbf{T}(x, y) = (2x + yi, x + 2y)$. Determine whether $\mathbf{T}$ is normal, self-adjoint, or neither.

39. Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$. Show that A is orthogonally equivalent to a diagonal matrix, and find an orthogonal matrix P and a diagonal matrix D such that $P^t A P = D$.