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Unital designs with blocking sets

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ABSTRACT

A unital 2-(28, 4, 1) design has 28 points, each block has size 4 and every pair of points is on exactly one block. A blocking set in a design is a subset of the point set with the property that every block intersects the blocking set nontrivially but no block is contained in the blocking set. In this work, we classify the unital 2-(28, 4, 1) designs with blocking sets. We find 68,806 unitals with a blocking set. Of these, 68,484 have a trivial automorphism group.

1. Introduction and statement of results

A *finite incidence geometry* is a pair $(P, \mathcal{L})$. Here, P is a finite set whose elements are called *points*, and $\mathcal{L}$ is a set of subsets of P called *lines*, such that

1. each subset $\ell \in \mathcal{L}$ has at least two points, and
2. each pair of points is on at most one block.

A *linear space* is an incidence geometry $(P, \mathcal{L})$ such that each pair of points is on exactly one line. We refer to [3] for more details on linear spaces.

A 2-(v, k, λ) design is a pair $(P, \mathcal{L})$ such that P is a set of size v, $\mathcal{L}$ is a set of k-subsets of P, and for each pair of points in P there are exactly λ elements in $\mathcal{L}$ that contain both points. Thus, a 2-$(v, k, 1)$ design is a linear space with all lines of size k. Often, the elements of $\mathcal{L}$ in a design are called *blocks* (and the design is called a *block system*). A 2-$(v, k, 1)$ design is also known as a *Steiner* 2*-design*.

An incidence geometry $(P, \mathcal{L})$ has a *blocking set* if there exists a subset $B \subset P$ with the following properties:

1. $B \cap \ell \neq \emptyset$ for all lines $\ell \in \mathcal{L}$ and
2. $\ell \subset B$ for no line $\ell \in \mathcal{L}$.

For $n \geq 2$, a 2-$(n^3+1, n+1, 1)$ design is called a *unital design* of order n. In particular, a 2-(28, 4, 1) unital design consists of 28 points and 63 lines of size 4, so that each pair of points is on exactly one block. A *unital* is a set S of $n^3 + 1$ points in a projective plane $\pi = (P, \mathcal{L})$ of order n such that $|S \cap \ell| \in \{1, n + 1\}$ for all lines $\ell \in \mathcal{L}$.

A classical example of a unital is the Hermitian unital which is the set of absolute points of a unitary polarity in the desarguesian projective plane of order q^2, PG$(2, q^2)$. Another example is the Ree unital which is a design on $q^3 + 1$ points associated with the Ree group, see [10].

Units (i.e., unital designs that are embedded in projective planes) have received much attention. The unitals of order 3 were classified in [12]. The work [13] focuses on unitals in planes of order 16. The unitals of order 4 in the desarguesian projective plane of order 16 have been classified in [1]. The book [2] is dedicated to the study of unitals.

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Brouwer [6] was the first to show the difference between unitals and unital designs. A unital is a unital design but not conversely. The classification of *all* unital designs of order 3 seems to be beyond reach at the moment. Penttila and Royle in [12] put it this way:

> *The problem of constructing all* 2-(28, 4, 1) *designs seems to be infeasible at this stage (though possibly not by much).*

The present work is a contribution to the classification of unital 2-(28, 4, 1) designs. Brouwer [6] found 138 unital designs of order 3. Seven more unital designs were found in [11]. In [9], unital designs with non-trivial automorphism group were classified. There are exactly 4466 such unital designs. Several unitals with trivial automorphism group were found in [5]. In the present article, we make the additional assumption that there exists a *blocking set*. Thus, we classify all unitals of order 3 admitting a blocking set. We find 68,806 such unitals, of which 68,484 have a trivial automorphism group.

The paper is organized in the following way. In Section 2, we recall some basic definition of tactical decomposition. In Section 3, the notion of a blocking set is related to that of discrepancy. In Section 4, we consider all of the possible decomposition schemes of unitals admitting a blocking set. The classification procedure is described in Section 5.

2. Tactical decomposition

In this section, we recall some basic definitions of tactical decompositions. For further reference, see [7]. In the following, we will consider an incidence geometry $\mathcal{X} = (P, \mathcal{L})$. For $p \in P$, we define

$$(p) = \{\ell \in \mathcal{L} \mid p \in \ell\}$$

the *pencil* of lines through p.

A *decomposition* of an incidence geometry $\mathcal{X} = (P, \mathcal{L})$ is a pair $(\mathcal{R}, \mathcal{C})$ of ordered partitions where $\mathcal{R} = (R_1, R_2, \ldots, R_m)$ is a partition of points and $\mathcal{C} = (C_1, C_2, \ldots, C_n)$ is a partition of blocks. For each $i = 1, 2, \ldots, m$ and each $j = 1, 2, \ldots, n$, let $r_{i,j} = |\{B \in C_j \mid p \in B\}|$ with $p \in R_i$ fixed. Let $c_{i,j} = |\{p \in R_i \mid p \in B\}|$, for $B \in C_j$ fixed.

A decomposition $(\mathcal{R}, \mathcal{C})$ of $\mathcal{X}$ is said to be *point tactical* if the number $r_{i,j}$ is independent of the choice $p \in R_i$ for each i and for each j. It is said to be *block tactical* (or *line tactical*) if for each i and for each j, the number $c_{i,j}$ is independent of the choice $B \in C_j$. A point tactical and a block tactical decomposition is simply said to be *tactical* decomposition with respect to $\mathcal{X}$.

Given a point tactical decomposition $(\mathcal{R}, \mathcal{C})$ of an incidence geometry, we define the *decomposition scheme* in the following way. These numbers are $a_i = |R_i|$ for $1 \le i \le m$, $b_j = |C_j|$ for each $1 \le j \le n$ together with the integers $r_{i,j}$ and $c_{i,j}$. In particular, if $\mathcal{R} = \{R_1, \ldots, R_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$, we use a form as in (1) to describe such a scheme.

$$\begin{array}{c|cccc} \rightarrow & b_1 & b_2 & \cdots & b_n \\ \hline a_1 & r_{1,1} & r_{1,2} & \cdots & r_{1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & r_{m,1} & r_{m,2} & \cdots & r_{m,n} \end{array}. \tag{1}$$

Note that the horizontal arrow in the upper left corner is indicating that this scheme describes a point tactical decomposition.

In a similar way, if $(\mathcal{R}, \mathcal{C})$ is a block tactical decomposition, then the form in (2) describes the corresponding decomposition scheme. Notice that the horizontal arrow is replaced by a vertical one. This indicates that we have a block tactical decomposition.

$$\begin{array}{c|cccc} \downarrow & b_1 & b_2 & \cdots & b_n \\ \hline a_1 & c_{1,1} & c_{1,2} & \cdots & c_{1,n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_m & c_{m,1} & c_{m,2} & \cdots & c_{m,n} \end{array}. \tag{2}$$

3. Discrepancy and blocking sets

We recall the notion of discrepancy, here only applied to finite incidence geometries, see [4].

Let $(P, \mathcal{L})$ be a finite incidence structure. Let $\Pi = (A, B)$ be a non-trivial partition of the point set P with two classes (i.e., with $P = A \cup B$, with $A \cap B = \emptyset$ and with $1 < |A| < |P|$). For each line $\ell \in \mathcal{L}$, one calculates the values $|A \cap \ell|$ and $|B \cap \ell|$ and takes the positive difference. The maximum of all such numbers is

$$u(\Pi) = \max_{L} \{||A \cap \ell| - |B \cap \ell||\}.$$

The *discrepancy* of $(P, \mathcal{L})$ is the smallest value of $u(\Pi)$ when considering all non-trivial partitions of P with two classes:

$$\Delta = \min_{\Pi} u(\Pi).$$

In the present work, we are interested in 2-(28, 4, 1) designs. Since all blocks have length 4 in these designs, only the following three possibilities for the value of u exist: $4 - 0 = 4$, $3 - 1 = 2$ and $2 - 2 = 0$. It follows that the discrepancy is one of the values 4, 2 or 0. The case $\Delta = 0$ does not arise:

Proposition 1. *The discrepancy Δ of a* 2-(28, 4, 1) *design is either* 2 *or* 4*.*

Proof. To prove this, we have to exclude $\Delta = 0$. In such case, there would exist a partition $\Pi = (A, B)$ such that the induced subgeometries on A and on B only have 2-blocks. These subgeometries are also linear spaces. Therefore, the number of 2-lines must be of the form $\binom{w}{2}$, $w \in \mathbb{Z}$. But the number of blocks is 63 and is not of this form. □

The notion of a blocking set (see Section 1) is closely related to the discrepancy:

Proposition 2. *A* 2-(28, 4, 1) *design has a blocking set if and only if the discrepancy is* 2*.*

Proof. Let B be a blocking set for the 2-(28, 4, 1) design. Then B and $A = P \setminus B$ define a partition $\Pi = (A, B)$ of P, where all differences are $3 - 1 = 2$ or $2 - 2 = 0$. Therefore, the maximum value of these differences is $u(\Pi) \in \{0, 2\}$. But the case of $u(\Pi) = 0$ is not possible as this would imply $\Delta = 0$, a contradiction. The minimum $u(\Pi)$ over all partitions Π is 2. So we get $\Delta = 2$.

Conversely, if $\Delta = 2$, then there is a partition $\Pi = (A, B)$ with $u(\Pi) = 2$. Therefore we can take A (and also B) as blocking set. □

In this paper, we classify all 2-(28, 4, 1) unital designs with blocking sets. According to Proposition 2, this means we classify all 2-(28, 4, 1) unital designs with discrepancy equal to 2.

4. The possible decompositions

We make now the assumption that the unitals have a blocking set, and we determine the possible tactical decompositions.

Lemma 3. *Let $(P, \mathcal{L})$ be a* 2-(28, 4, 1) *unital design. The column tactical decomposition*

↓	21	3	39
12	3	2	1
16	1	2	3

does not arise.

Proof. Let (Π, Ψ) be the decomposition that gives rise to the decomposition scheme. That is, we have $\Pi = (A, B)$ a partition of P with $|A| = 12$ and $|B| = 16$. Also $\Psi = (\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3)$ a partition of $\mathcal{L}$ with $|\mathcal{L}_1| = 21$, $|\mathcal{L}_2| = 3$ and $|\mathcal{L}_3| = 39$. For $p \in A$, let $x_i = |(p) \cap \mathcal{L}_i|$ for $i = 1, 2, 3$. Also, for $p \in B$, let $y_i = |(p) \cap \mathcal{L}_i|$ for $i = 1, 2, 3$. Then

$$2x_1 + x_2 = 11,$$
$$x_1 + 2x_2 + 3x_3 = 16,$$
$$x_1 + x_2 + x_3 = 9,$$
$$3y_1 + 2y_2 + y_3 = 12,$$
$$y_2 + 2y_3 = 15,$$
$$y_1 + y_2 + y_3 = 9$$

which yields

$$(x_1, x_2, x_3) \in \{(5, 1, 3),\ (4, 3, 2)\}, \qquad (y_1, y_2, y_3) \in \{(1, 1, 7),\ (0, 3, 6)\}.$$

Let w_1, w_2 be the number of points of A of the first two types, and let w_3, w_4 be the number of points of B of the latter two types. Then

$$5w_1 + 12w_2 + w_3 \le 63,$$
$$3w_1 + 6w_2 + 7w_3 + 18w_4 \le 117,$$
$$w_1 + w_2 = 12,$$
$$w_3 + w_4 = 16,$$
$$w_1 + w_2 + w_3 + w_4 = 28.$$

The first and third condition imply $(w_1, w_2) = (12, 0)$. Thus the first condition becomes $w_3 \le 3$, which forces $w_4 \ge 13$. But then the second condition is violated. □

Lemma 4. *Let $(P, \mathcal{L})$ be a* 2-(28, 4, 1) *unital design with column tactical decomposition*

↓	24	6	33
13	3	2	1
15	1	2	3

.

The decomposition can be refined uniquely to the row tactical decomposition

$\rightarrow$	24	6	33
7	6	0	3
6	5	2	2
9	2	0	7
6	1	2	6

.

Proof. We use notation similar to that of the proof of Lemma 3. Counting arguments show that

$$(x_1, x_2, x_3) \in \{(6, 0, 3),\ (5, 2, 2),\ (4, 4, 1),\ (3, 6, 0)\}$$

and

$$(y_1, y_2, y_3) \in \{(2, 0, 7),\ (1, 2, 6),\ (0, 4, 5)\}.$$

Let w_1, w_2, w_3, w_4 be the number of points of A of the first 4 types. Let w_5, w_6, w_7 be the number of points of B of the latter 3 types. Then

$$\begin{aligned} &w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 28, \\ &w_2 + 6w_3 + 15w_4 + w_6 + 6w_7 \le 15, \\ &6w_1 + 5w_2 + 4w_3 + 3w_4 = 72, \\ &2w_2 + 4w_3 + 6w_4 = 12, \\ &2w_5 + w_6 = 24. \end{aligned}$$

The only solution to this system is

$$(w_1, \ldots, w_7) = (7, 6, 0, 0, 9, 6, 0).$$

Thus we get the unique refinement as claimed. □

Theorem 5. *Each* 2-(28, 4, 1) *unital design with a blocking set admits one of the following decomposition schemes*

Type D:

$\rightarrow$	7	28	28
14	1	6	2
14	1	2	6

,

Type E:

$\rightarrow$	24	6	33
7	6	0	3
6	5	2	2
9	2	0	7
6	1	2	6

. (3)

The first decomposition (type D) is tactical while the second one (type E) is only point tactical. It is a refinement of the column tactical decomposition

$\downarrow$	24	6	33
13	3	2	1
15	1	2	3

. (4)

Proof. Let $\Pi = (A, B)$ be a partition of the point set P with $\Delta = 2 = u(\Pi)$. Let $|A| = a$ and $|B| = b$. Then, the partition Π has the following block tactical decomposition.

$\downarrow$	x	y	z
a	3	2	1
b	1	2	3

.

Since $a + b = 28$, we can assume without loss of generality that $a \le 14$. Moreover,

$$x + y + z = 63, \tag{5}$$

since there are 63 lines total. Counting flags (incident point–line pairs) inside A, we get

$$3x + 2y + z = 9a. \tag{6}$$

This is together with (5) implies

$$2x + y = 9a - 63. \tag{7}$$

Counting pairs of points $(p, q) \in A \times B$, we get

$$3x + 4y + 3z = ab. \tag{8}$$

Using (5), we find

$$y = ab - 189. \tag{9}$$

Under the assumption $a \leq 14$, this implies

$$(a, b, y) \in \{(12, 16, 3),\ (13, 15, 6),\ (14, 14, 7)\}. \tag{10}$$

Now, x can be computed by using (7) and z follows by using (5). In the three cases, we get

$$(x, z) \in \{(21, 39),\ (24, 33),\ (28, 28)\}. \tag{11}$$

Therefore we arrive at the following three block tactical decompositions:

$$\begin{array}{c|ccc} \downarrow & 21 & 3 & 39 \\ \hline 12 & 3 & 2 & 1 \\ 16 & 1 & 2 & 3 \end{array}, \qquad \begin{array}{c|ccc} \downarrow & 24 & 6 & 33 \\ \hline 13 & 3 & 2 & 1 \\ 15 & 1 & 2 & 3 \end{array}, \quad \text{and} \quad \begin{array}{c|ccc} \downarrow & 28 & 7 & 28 \\ \hline 14 & 3 & 2 & 1 \\ 14 & 1 & 2 & 3 \end{array}.$$

The first decomposition is ruled out by Lemma 3. The second decomposition refines to the decomposition of type E by Lemma 4.

In the third case, it is easy to show that the decomposition is tactical (i.e., both point tactical and block tactical). This is the decomposition of type D. □

The decomposition of type D is suitable for computer classification. On the other hand, the decomposition of type E is still too coarse, so we wish to refine it. Refining the decomposition further would give too many cases. So, we concentrate on the subgeometry with the 6-block domain (i.e., the lines $\ell \in \mathcal{L}$ with $|A \cap \ell| = |B \cap \ell| = 2$). In particular, we consider the partial decomposition

$$\begin{array}{c|c} \rightarrow & 6 \\ \hline 6 & 2 \\ 6 & 2 \end{array}$$

which is tactical. The upper part

$$\begin{array}{c|c} \rightarrow & 6 \\ \hline 6 & 2 \end{array}$$

generates either a 6-cycle or two 3-cycles. The same holds for the lower part. This gives four cases:

1. Two 3-cycles in the upper part and a 6-cycle in the lower part,
2. 6 cycle in the upper part and two 3-cycles below,
3. two 6-cycles, one in the upper part and one in the lower part,
4. two 3-cycles in the upper part and two 3-cycles in the lower part.

The fourth case is easily seen to be impossible. The first three cases can be described by

$$\overset{1.}{\begin{array}{c|cc} \rightarrow & 3 & 3 \\ \hline 3 & 2 & 0 \\ 3 & 0 & 2 \\ \hline 6 & 1 & 1 \end{array}}, \qquad \overset{2.}{\begin{array}{c|cc} \rightarrow & 3 & 3 \\ \hline 6 & 1 & 1 \\ \hline 3 & 2 & 0 \\ 3 & 0 & 2 \end{array}}, \qquad \overset{3.}{\begin{array}{c|c} \rightarrow & 6 \\ \hline 3 & 2 \\ 3 & 2 \\ \hline 3 & 2 \\ 3 & 2 \end{array}}.$$

These schemes can be included in the decomposition scheme of type E as described in Theorem 6.

Theorem 6. *The unitals generated from the scheme of type E are exactly those which are generated from the schemes of type F, type G, and type H. These three types are presented in (12).*

$$\overset{\text{Type F}}{\begin{array}{c|cccc} \rightarrow & 3 & 3 & 24 & 33 \\ \hline 7 & 0 & 0 & 6 & 3 \\ 3 & 2 & 0 & 5 & 2 \\ 3 & 0 & 2 & 5 & 2 \\ \hline 6 & 1 & 1 & 1 & 6 \\ 9 & 0 & 0 & 2 & 7 \end{array}}, \qquad \overset{\text{Type G}}{\begin{array}{c|cccc} \rightarrow & 3 & 3 & 24 & 33 \\ \hline 7 & 0 & 0 & 6 & 3 \\ 6 & 1 & 1 & 5 & 2 \\ \hline 3 & 2 & 0 & 1 & 6 \\ 3 & 0 & 2 & 1 & 6 \\ 9 & 0 & 0 & 2 & 7 \end{array}}, \quad \text{and} \quad \overset{\text{Type H}}{\begin{array}{c|ccc} \rightarrow & 6 & 24 & 33 \\ \hline 7 & 0 & 6 & 3 \\ 3 & 2 & 5 & 2 \\ 3 & 2 & 5 & 2 \\ \hline 3 & 2 & 1 & 6 \\ 3 & 2 & 1 & 6 \\ 9 & 0 & 2 & 7 \end{array}}. \tag{12}$$

5. Classification of the unital designs

In this section, we describe the classification procedure for unital designs admitting a blocking set.

Table 1
Computing the list of starters for decomposition schemes of types D, F, G, and H.

Type	Depth n	Number of generated cases	Number of starters
D	14	787	787
F	13	59	32
G	13	13,365	191
H	13	13,369	191

By the results of Section 4, this is equivalent to classifying the geometries from the decomposition schemes of type D (as in (3)), and of types F, G, and H (as in (12)). We assume familiarity with the basics of classification algorithms of incidence geometries. A detailed introduction to such algorithms can be found in [8].

We begin by classifying geometries satisfying these decompositions partially. By this, we mean that we classify the *partial incidence matrices* of geometries admitting a decomposition. A partial incidence matrix is a matrix whose first n rows satisfy the conditions imposed by the decomposition as well as by the geometry, but whose remaining rows are all zero. The idea is to fill the remaining rows of the incidence matrix later. We say that n is the depth of the partial incidence matrix. The parameter n is chosen after some experimentation, and it varies from decomposition to decomposition. The goal is to first classify all partial incidence matrices at depth n under the group fixing the decomposition scheme *and* fixing the first n rows setwise. For type D we choose $n = 14$, and for all other decompositions schemes we choose $n = 13$. We represent each orbit of partial incidence matrices by its *canonical representative.* This is just the matrix that is least in the lexicographical ordering of all matrices in its isomorphism class.

In the next step, we forget about the decomposition scheme. That is, we classify the partial incidence matrices under the group that fixes the first n rows but otherwise is allowed to move points across the boundaries of the decomposition. This eliminates copies of geometries that differ only by a rearrangement of the decomposition classes. For instance, in the case of type F we have two 3-cycles which can be permuted and hence result isomorphic copies. Therefore, we do an isomorphism test at line 14 (type D) and at line 13 (types F, G, and H) under the group that fixes the first 14 rows setwise. The resulting list of cases (for each decomposition scheme type) are called the *list of starters* (at depth n). The number of isomorphism types of partial incidence matrices at depth n, and the number of starters at depth n is presented in Table 1.

For each starter at depth n, we run a computer search to generate the partial incidence matrices at some depth m with $n < m$. Here, $m = 18$ for type D and $m = 19$ for the remaining types. In order to find the full incidence matrices of the geometries that we are looking for, we need to complete the partial incidence matrices at depth m to depth 28. This can be formulated as a problem of finding all cliques in a certain graph: For each isomorphism type of starter S at depth m, we define a graph Γ_S. The vertices of Γ_S are the possibilities for the missing rows. These are the 0/1 vectors that could potentially be used to fill in additional rows of the partial incidence matrix. Of course, these 0/1 vectors have to satisfy the conditions that result from the decomposition (as well as from the fact that we create a linear space). Two vertices of Γ_S are connected if the corresponding vectors can both be chosen *simultaneously* to extend the partial incidence matrix. This requires testing if the rows have exactly one position where both have an entry one. Also, it requires testing if the conditions on the column sums are satisfied.

For each starter, we present the number of cliques in the second column of Table 2 (type D), Table 3 (type F), Table 4 (type G), and Table 5 (type H).

Once the incidence matrices have been filled to depth 28 using the cliques of size $28 - m$ from the graph Γ, we perform an additional round of isomorphism testing. This produces the classification of all geometries for a given scheme with a given starter at depth n (in the third column). After that we consider the geometries that are obtained from all starters and perform another round of isomorph rejection, this time disregarding the decomposition schemes. The numbers that result from this are listed at the bottom as "merged". The result of this procedure is presented in Theorem 7.

Theorem 7. *The decomposition scheme of types D, F, G, and H generate the numbers of (pairwise non-isomorphic) unitals presented in Table* 6*.*

Considering the unitals from all decomposition schemes together, and performing another round of isomorphism testing leads to the following result:

Theorem 8. *The number of (pairwise non-isomorphic) unitals of order* 3 *with discrepancy* 2 *(i.e., having a blocking set) is* 68,806*. The distribution of automorphism group orders is*

$$\{192^2, 48^7, 24^2, 16^2, 12^7, 8^{19}, 7, 6^3, 4^{87}, 3^{102}, 2^{90}, 1^{68484}\}.$$

6. Note added in the proof

In work that was done after this paper was first submitted, we checked our results against the list of unital designs with non-trivial automorphism group [9]. The number of unital designs with non-trivial automorphism group is 4466. We tested which of these had a blocking set and found that exactly 322 did. This agrees with our number of $68{,}806 - 68{,}484 = 322$ unital designs with blocking set and non-trivial automorphism group from Theorem 8.

Table 2
Unitals of type D, generated from the list of starters at line 14.

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
1	18	1	292	12	2	531	16	1
11	36	13	294	8	1	532	8	2
12	114	19	295	4	1	533	32	8
14	16	1	297	8	1	536	4	1
16	8	1	299	8	1	537	16	1
21	8	2	301	12	2	539	8	2
23	4	1	303	66	4	540	4	1
24	2	1	305	4	1	542	14	4
26	8	1	306	8	3	543	16	1
30	2	1	307	24	5	545	20	2
32	16	1	308	40	11	546	8	1
35	8	3	310	32	2	547	16	1
39	352	10	312	40	7	550	32	7
40	32	3	313	24	6	551	14	3
41	4	1	314	16	1	555	4	1
42	4	1	315	4	1	559	8	1
50	8	1	316	8	1	561	4	1
51	8	2	317	8	1	562	16	4
52	8	1	318	20	2	564	20	3
55	12	2	319	6	2	566	24	4
59	8	2	321	4	1	567	8	1
61	10	2	323	4	1	568	24	3
67	4	1	325	4	1	569	20	2
69	24	1	326	8	1	570	16	2
72	80	4	332	2	1	572	10	2
73	80	6	333	2	1	573	32	4
76	4	1	335	18	4	575	8	1
77	4	1	340	24	3	577	46	7
79	48	3	342	8	1	580	8	1
80	152	42	344	16	1	583	12	2
89	40	2	345	2	1	585	80	13
90	6	2	347	8	1	586	16	1
93	4	1	348	20	2	588	8	2
96	4	1	349	4	1	589	4	1
98	24	1	350	4	1	590	12	2
99	16	1	351	8	3	593	8	2
101	16	2	354	16	1	595	16	1
102	4	1	355	8	2	597	20	2
105	26	3	358	12	2	600	16	2
109	2	1	359	16	1	601	8	2
112	8	1	360	16	2	602	8	1
113	28	4	362	16	3	603	8	2
114	4	1	363	10	2	604	4	1
115	4	1	364	8	2	611	8	2
116	4	2	368	2	1	614	24	2
117	6	2	369	10	2	615	8	1
118	40	7	370	8	1	617	4	1
119	16	1	372	8	1	618	24	2
120	24	3	374	4	1	620	8	1
121	20	2	375	8	1	622	4	1
122	44	5	377	6	2	623	4	1
123	8	2	378	8	1	624	12	2
126	60	4	379	4	1	629	2	1
129	2	1	380	10	3	632	8	4
130	8	1	381	2	1	637	8	4
131	6	2	382	24	3	638	10	2
132	2	1	383	8	1	640	22	5
133	4	1	384	8	1	641	8	1
137	16	1	386	12	2	646	16	3
138	8	2	387	8	1	648	4	1
139	4	1	390	8	1	651	8	2
140	4	1	391	16	2	654	8	2
145	4	1	392	2	1	659	4	1
146	16	2	393	8	1	661	8	1
147	4	1	394	48	9	662	4	1
149	32	1	395	28	4	665	8	1
150	12	2	396	40	6	669	12	2
152	2	1	397	2	1	670	38	6

Table 2 (*continued*)

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
154	4	1	399	28	3	672	40	2
157	4	1	400	12	2	674	32	1
158	16	2	401	4	1	675	8	1
161	4	1	403	16	1	676	4	1
162	22	3	405	8	2	677	8	1
166	16	4	409	24	3	678	34	3
168	16	1	412	6	2	680	12	2
170	4	1	414	8	1	682	8	1
171	28	3	415	4	1	683	6	2
172	12	2	416	16	1	684	20	1
174	24	3	417	32	6	685	4	1
176	4	1	423	4	1	686	6	2
177	8	2	424	10	2	687	16	3
179	50	8	425	4	1	689	16	1
180	2	1	426	24	2	691	8	1
183	36	2	427	12	2	692	8	2
184	16	1	428	2	1	694	20	2
185	40	2	430	2	1	695	8	1
186	4	1	431	4	1	696	10	2
187	4	1	433	16	1	697	8	2
192	144	9	434	2	1	699	24	2
193	8	1	436	16	1	700	24	2
195	12	2	440	20	1	702	20	2
196	2	1	442	24	5	703	16	2
197	8	1	448	4	1	705	4	1
198	4	1	452	4	1	706	4	1
202	44	4	453	8	1	707	12	2
203	16	3	455	8	1	708	40	5
204	20	2	458	4	1	709	8	2
206	16	2	459	24	2	710	4	1
207	68	6	460	16	2	712	24	3
208	8	1	461	4	2	713	12	2
211	20	2	462	44	10	714	4	1
212	4	1	464	14	3	715	8	1
213	16	1	465	16	3	717	12	2
214	16	1	466	8	1	718	12	3
216	4	1	468	4	1	719	14	1
217	56	3	470	16	1	720	808	72
218	8	1	472	8	1	721	8	1
220	16	2	473	8	1	723	4	1
222	2	1	474	32	7	724	398	32
223	4	1	475	2	1	726	6	2
225	16	1	476	8	1	728	8	1
226	16	1	478	8	2	731	12	2
227	12	2	481	4	1	732	4	1
229	2	1	484	4	1	735	8	1
231	8	1	485	4	1	736	4	1
232	12	2	486	10	3	737	4	1
237	10	2	489	16	2	738	18	1
242	24	2	491	2	1	740	36	3
245	4	1	492	16	4	741	4	1
246	16	1	494	16	2	742	34	5
247	4	1	495	10	2	743	8	1
248	16	7	497	16	1	744	18	2
249	18	8	499	6	2	745	4	1
253	16	1	500	16	1	746	12	2
254	8	1	501	12	2	747	2	1
255	8	1	503	24	6	749	32	4
256	8	1	504	20	2	752	240	8
257	8	1	505	26	3	754	16	1
258	8	1	506	4	1	757	2	1
259	4	2	508	24	2	759	2	1
260	24	2	515	4	1	761	24	2
265	8	1	517	4	1	767	16	2
276	8	1	519	4	1	769	4	1
277	4	1	520	22	5	771	4	1
279	12	2	521	4	1	772	22	5
280	32	2	522	4	2	773	24	3
283	6	2	523	4	1	774	8	1
284	12	2	526	16	4	776	8	1

(*continued on next page*)

Table 2 (*continued*)

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
288	12	2	527	4	1	778	4	1
290	32	2	528	16	1	780	6	2
291	4	1	530	24	2	783	8	1
						Total:		1015
						Merged:		436

Table 3
Unitals of type F, generated from the list of starters at line 13.

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
1	4106	534	12	2362	206	23	2000	252
2	4526	557	13	4600	571	24	4550	277
3	4340	525	14	4312	533	25	4470	561
4	4268	554	15	1354	106	26	4514	566
5	4494	571	16	2030	277	27	2232	279
6	4110	533	17	1958	261	28	2378	276
7	4190	546	18	4170	276	29	4286	283
8	4196	563	19	4418	277	30	4292	549
9	4162	525	20	4502	276	31	4678	561
10	4460	591	21	4350	540	32	4456	554
11	4616	596	22	2220	265			
						Total:		13,841
						Merged:		13,819

Table 4
Unitals of type G, generated from the list of starters at line 13.

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
1	499	86	65	463	95	129	442	71
2	489	95	66	216	38	130	467	83
3	537	90	67	253	41	131	599	106
4	480	94	68	475	89	132	674	106
5	657	107	69	662	103	133	524	79
6	454	91	70	578	98	134	449	85
7	561	94	71	488	71	135	396	81
8	382	79	72	238	36	136	509	45
9	627	95	73	542	48	137	525	81
10	561	90	74	556	94	138	538	76
11	468	84	75	503	48	139	579	87
12	524	92	76	545	90	140	466	76
13	567	92	77	431	39	141	448	84
14	562	98	78	506	89	142	523	41
15	568	89	79	229	21	143	385	71
16	429	78	80	422	39	144	477	81
17	439	88	81	286	50	145	414	63
18	546	91	82	298	45	146	412	72
19	425	71	83	403	79	147	427	65
20	408	77	84	533	90	148	414	67
21	431	84	85	488	83	149	480	38
22	500	80	86	567	90	150	422	54
23	614	93	87	586	96	151	555	86
24	648	116	88	499	98	152	688	98
25	529	96	89	401	81	153	399	67
26	616	96	90	514	106	154	512	77
27	476	92	91	606	100	155	354	69
28	516	93	92	573	94	156	548	39
29	493	91	93	476	89	157	481	77
30	433	87	94	437	82	158	517	79
31	477	84	95	403	76	159	522	79
32	502	93	96	396	81	160	542	81
33	382	81	97	479	85	161	447	30
34	547	87	98	478	94	162	528	71
35	537	95	99	574	92	163	523	85

Table 4 (*continued*)

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
36	389	74	100	755	60	164	404	70
37	571	101	101	255	37	165	483	37
38	516	81	102	700	119	166	399	72
39	633	97	103	628	57	167	474	75
40	642	106	104	193	38	168	570	89
41	517	103	105	570	99	169	556	85
42	490	81	106	644	104	170	442	38
43	521	94	107	435	81	171	494	67
44	454	78	108	530	90	172	430	68
45	478	91	109	501	90	173	299	55
46	445	85	110	276	57	174	491	69
47	622	93	111	381	65	175	611	86
48	418	82	112	495	103	176	666	89
49	565	90	113	288	46	177	477	67
50	483	79	114	268	52	178	466	72
51	392	77	115	590	89	179	534	76
52	502	91	116	665	106	180	96	21
53	512	91	117	538	84	181	514	70
54	513	97	118	552	115	182	531	80
55	510	92	119	487	94	183	354	59
56	498	95	120	564	98	184	395	65
57	587	103	121	482	89	185	585	90
58	390	586	122	468	80	186	486	69
59	214	39	123	545	84	187	256	20
60	278	44	124	504	81	188	324	23
61	516	93	125	505	86	189	374	47
62	471	88	126	591	46	190	390	52
63	564	97	127	466	86	191	226	26
64	553	115	128	451	88			
						Total:		15,114
						Merged:		14,901

Table 5
Unitals of type H, generated from the list of starters at line 13.

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
1	3660	258	65	1059	76	129	3519	245
2	3266	239	66	1354	89	130	3568	253
3	1316	82	67	3602	261	131	3135	223
4	3284	237	68	3551	262	132	3303	243
5	954	78	69	3027	217	133	3430	133
6	1041	72	70	3747	276	134	3222	126
7	3475	249	71	3490	247	135	2946	113
8	3204	229	72	3737	260	136	3792	131
9	3607	253	73	1533	114	137	3208	123
10	4051	296	74	3700	266	138	3808	133
11	3654	250	75	3467	246	139	3994	275
12	3839	264	76	1807	128	140	3053	205
13	3678	268	77	1555	107	141	2991	204
14	3418	248	78	3740	269	142	3170	211
15	4153	293	79	3311	119	143	3155	214
16	3656	258	80	3656	127	144	3333	204
17	3327	237	81	1071	73	145	3168	220
18	3158	229	82	965	39	146	1694	113
19	4209	277	83	3495	252	147	1433	92
20	3400	250	84	3330	256	148	2700	187
21	3738	258	85	3437	234	149	1663	107
22	3453	250	86	3040	229	150	1722	117
23	3779	258	87	3351	235	151	1038	70
24	3579	246	88	3388	236	152	1564	102
25	3068	217	89	3378	248	153	2498	175
26	3660	253	90	2887	216	154	3164	211
27	3118	223	91	3491	257	155	3199	207
28	3295	240	92	3697	254	156	3219	221
29	3197	239	93	3143	219	157	2880	196
30	3591	248	94	3720	249	158	2875	193
31	3618	269	95	2967	215	159	3200	205
32	3152	225	96	3259	240	160	3225	218

(*continued on next page*)

Table 5 (*continued*)

Nr.	gen.	isot.	Nr.	gen.	isot.	Nr.	gen.	isot.
33	3336	251	97	2941	207	161	3484	242
34	3221	223	98	3517	125	162	3248	224
35	3057	233	99	3061	230	163	2813	207
36	3726	265	100	3151	124	164	2913	203
37	3676	260	101	3308	118	165	3036	207
38	3426	250	102	3304	121	166	2779	181
39	3476	246	103	3056	211	167	2992	207
40	3317	243	104	3496	244	168	3273	211
41	3457	247	105	3334	237	169	3211	213
42	2995	223	106	3584	244	170	2860	200
43	3607	252	107	3227	235	171	2852	194
44	3037	226	108	3358	236	172	2995	209
45	3310	245	109	3456	236	173	2795	182
46	2993	229	110	3718	259	174	3423	226
47	3488	238	111	3388	244	175	3053	215
48	3446	238	112	3584	248	176	2897	196
49	3493	243	113	3370	236	177	2226	163
50	3511	246	114	3892	258	178	2696	175
51	3564	261	115	3400	244	179	2452	171
52	3450	244	116	3198	220	180	2626	171
53	3268	243	117	3134	234	181	2663	173
54	3537	256	118	3543	245	182	2918	204
55	3140	213	119	3439	245	183	2716	192
56	3306	231	120	3316	226	184	3125	203
57	3710	262	121	3879	280	185	1119	76
58	3517	260	122	3567	248	186	2561	186
59	3510	251	123	3091	218	187	974	63
60	3856	273	124	2880	212	188	1632	106
61	3673	257	125	3224	250	189	2268	140
62	4183	294	126	3325	227	190	888	63
63	3274	235	127	3711	255	191	816	51
64	1306	94	128	3498	249			
						Total:		40,031
						Merged:		39,886

Table 6
The generated unitals from the decomposition of types D, F, G, and H. The corresponding distributions of automorphism group orders (ago's) are presented in the third column.

Type	Solutions	Distribution of ago's
D	436	$\{1^{225}, 2^{78}, 3^{3}, 4^{87}, 6^{3}, 7, 8^{19}, 12^{7}, 16^{2}, 24^{2}, 48^{7}, 192^{2}\}$
F	13,819	$\{1^{13,770}, 2^{4}, 3^{45}\}$
G	14,901	$\{1^{14,846}, 2^{1}, 3^{54}\}$
H	39,886	$\{1^{39,876}, 2^{7}, 3^{3}\}$
Total:	69,042	

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