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Isomorph-rejection: Theory and an application

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ABSTRACT

We describe, in some detail, classification algorithms that are suitable for several common searching tasks in combinatorics. We first consider the theory of two orderly generation algorithms. The first is due to Faradžev and Read (independently), while the second is due to McKay. After that, we use the later one in the classification of $\{0,1\}$ matrices with v rows and b columns (where $v, b \leq 12$) so that each row and each column has constant row sum and constant column sum, respectively. As a result, there are $317,579,783$ non-isomorphic 12×12 matrices over $\{0,1\}$ with rows and columns summing to 4.

Keywords: $\{0,1\}$-matrices, classification, Isomorphism classes, orderly generation.

INTRODUCTION

A *finite incidence structure* $\mathcal{X}$ is a pair $(P, \mathcal{B})$, where P (is an m-set) $= \{p_1, p_2, ..., p_m\}$ and $\mathcal{B}$ (is an n-set) $= \{B_1, B_2, ..., B_n\}$. Moreover, P is called a set of *points* and $\mathcal{B}$ is called a set of *blocks* (or *lines*) such that $B_i \subseteq P$ for $i = 1, 2, ..., n$.

Let $\mathcal{X}_1 = (P_1, \mathcal{B}_1)$ and $\mathcal{X}_2 = (P_2, \mathcal{B}_2)$ be two given incidence structures. A bijective map $\tau : P_1 \to P_2$ which maps $\mathcal{B}_1$ onto $\mathcal{B}_2$ is called an *isomorphism* between $\mathcal{X}_1$ and $\mathcal{X}_2$. Note that $\tau(B) = \{\tau(p) \mid p \in B\}$. If such a bijective map exists, we say that $\mathcal{X}_1$ is *isomorphic* to $\mathcal{X}_2$, denoted by $\mathcal{X}_1 \cong \mathcal{X}_2$. It is well known that isomorphism of incidence structures defines an equivalence relation. The equivalence classes are called *isomorphism classes*.

The *classification problem* of incidence structures is the problem of determining isomorphism classes of such structures. *Isomorph-rejection* techniques (cf. swift) are widely used in classification algorithms to solve such problems. One goal of such techniques is to produce a list of structures with no isomorphs. Another one is to avoid redundant work in the search for structures of interests.

One possible naive isomorph-rejection technique is to carry out an exhaustive generation procedure which produces all possible structures satisfying a given collection of properties. All newly encountered structures are stored in a *list* $\mathcal{L}$ during the search (generation procedure). A newly encountered structure x is tested for isomorphism against all stored structures in $\mathcal{L}$. If an isomorphic copy of x was found in the list $\mathcal{L}$, it is ignored. Otherwise, x is stored in $\mathcal{L}$ and the search goes on. In the literature, this method is called the *classical method*, see (Kaski & Óstergård, 2006; Read, 1978).

Another approach is called *orderly generation*, which was introduced independently by Read (1978) and Faradžev (1978). This method selects (in a special way) a representative from each isomorphism class. This representative is called *canonical*. The nodes of the search tree are tested, if they are in canonical form. The test is dependent on the definition of such a form. Applications for this method, of many, appear in (Brinkmann, 1996; Dinitz *et al.*, 1994; Meringer, 1999).

The focus of the presented paper is on generation by canonical augmentation. This method requires that a structure is generated in a canonical way and it is not necessarily being canonical itself. In this paper, the generation by canonical augmentation is described in some detail in our own language which differ from that presented in McKay (1998). We apply such an algorithm to classify a given class of incidence structures. As a result, a new classification result has been achieved.

Preliminary

In this section, we define some notations that will be needed throughout the rest of the paper. Let $M_{m,n}$ denote the set of all $m \times n$ $\{0,1\}$-matrices. For $A = (a_{ij}) \in M_{m,n}$ and $B = (b_{ij}) \in M_{p,q}$, we define the *Kronecker product*, denoted by $A \otimes B$, to be the block matrix in $M_{mp,nq}$ defined by

$$A \otimes B = \begin{bmatrix} a_{1,1}B & \dots & a_{1,n}B \\ \vdots & \ddots & \vdots \\ a_{m,1}B & \dots & a_{m,n}B \end{bmatrix}_{mp \times nq}$$

Let $\mathcal{R}$ and $\mathcal{C}$ be the sets of row and column indices, respectively. For a matrix $A \in M_{m,n}$, we define $\mathrm{Row_i(A)} = [\mathrm{a_{i,1}} \;\; \mathrm{a_{i,2}} \;\; \dots \;\; \mathrm{a_{i,n}}]$ to be the entries in the i^{th} row of A, define $\mathrm{row-sum_i(A)}$ to be the sum of entries in the $\mathrm{i^{th}}$ row of A, and define $\mathrm{row-supp(A)} = \{\mathrm{i} \mid \mathrm{row-sum_i(A)} \neq 0\}$, for $i \in \mathcal{R}$. In a similar way, $\mathrm{Col_j(A)}$, $\mathrm{col-sum_j(A)}$, and $\mathrm{Colsupp(A)}$ are defined for $j \in \mathcal{C}$. For $0 \leq l \leq m$, let $M^{(l)}_{m,n} = \{A \in M_{m,n} \,\big|\, |\mathrm{row-supp(A)}| = \mathrm{l}\}$. Clearly, we have $\cup_{l=0}^{m} M^{(l)}_{m,n} = M_{m,n}$.

For $i \in \mathcal{R}$, let E_i denote the matrix in $M_{m,1}$ with

$$Row_j(E_i) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{otherwise} \end{cases}$$

For $A \in M_{m,n}^{(l)}$ and $B \in M_{m,n}^{(l+1)}$, for $0 \leq l \leq m-1$, we say that B is an *extension* of A, or that A is a *pre-image* of B if $A = B - \left(E_i \otimes \mathrm{Row_i(B)}\right)$ for some $i \in \mathrm{row-supp(B)}$. In this case, we write $A \prec B$.

Let G be a group acting on a finite set X. For $x \in X$, we write x^G and G_x for the *orbit* and the *stabilizer*, respectively, of x under the action of G. Two objects x and y in X are (isomorphic) contained in the same G-orbit if there is a $g \in G$ so that $x^g = y$. In this case, it can be shown that $G_y = g^{-1}G_x g$. It is obvious that either $x^G = y^G$ or $x^G \cap y^G = \emptyset$. In particular, X/G (the set of all G-orbits on X) is a partition of X.

A *G-invariant set* is a set that is stabilized as a set under the action of G. If G has only one orbit on X (X itself), then G is *transitive* on X. An *orbit transversal* for G on X, denoted by $\mathcal{T}(G, X)$, is a set of elements in X so that for any $x \in X$, there exists $g \in G$ with $x^g \in \mathcal{T}(G, X)$, and for any $x, y \in \mathcal{T}(G, X)$, we have $x^G \cap y^G = \emptyset$.

A map $\varphi : X \rightarrow G$ with $x^{\varphi(x)} \cong x$, and $x \cong y$ implies $x^{\varphi(x)} = y^{\varphi(y)}$ for any $x, y \in X$ is called the *canonical labeling map*. Moreover, $x^{\varphi(x)}$ is called the *canonical form of* x with respect to φ. Such a mapping can be computed, for instance, by using the graph isomorphism program nauty. Further details can be found in [Al-Azemi, 2009; Leon, 1997; McKay, 1977; Royle, 1998; Spence, 1995).

Action on ordered pairs

Let G be a group acting on two finite sets X and Y. Let R be a G-invariant relation on the product $X \times Y$. A fundamental problem of classification algorithms of incidence structures is to construct an orbit transversal $\mathcal{T}(G, Y)$ given an orbit transversal $\mathcal{T}(G, X)$.

This problem is one of the main goals of the presented paper. We remark that the presented theory in this section is not new. McKay's paper (McKay, 1998), for instance, contains most of these ideas. However, we find that his paper is very condensed and therefore hard to read. As a result, we find it necessary to summarize these results regarding the solution of the above stated problem in our own language in some more detail. To solve such a problem, we consider another action on R to construct a transversal $\mathcal{T}(G, R)$ from which we construct the desired transversal, namely $\mathcal{T}(G, Y)$. Theorem is fundamental in solving this problem. A detailed proof can be found in (Al-Azemi, 2009). The same idea is emphesized in Theorem .

Let G act *coordinate-wise* on the product $X \times Y$, i.e., for $(x, y) \in X \times Y$ and $g \in G$, we have $(x, y)^g = (x^g, y^g)$. It follows that R is a union of G-orbits on pairs from $X \times Y$. Any G-orbit on pairs from $X \times Y$ which is contained in R is called a *flag orbit*.

For $x \in X$ and $y \in Y$, consider the projection maps $\pi_1 : R \to X$, defined by $(x, y) \mapsto x$, and $\pi_2 : R \to Y$, defined by $(x, y) \mapsto y$. For any $(x, y) \in R$, conclude that $\pi_1\big((x, y)^G\big) = x^G$ and $\pi_2\big((x, y)^G\big) = y^G$. These two orbits are called *shadow orbits*. We remark that a flag orbit determines two shadow orbits, but the converse is not true in general, see (Kerber, 1999) for further detail. We also define the *extension set* $\pi_1^{-1}(x) = \{(x, z) \in R\}$ and the *pre-image set* $\pi_2^{-1}(y) = \{(z, y) \in R\}$.

Theorem 1 *Let* $\boldsymbol{G}$ *be a group acting on two finite sets* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and let* $\boldsymbol{R} \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ *be a* $\boldsymbol{G}$*-invariant relation. Then, the following three sets are in canonical one-to-one correspondence.*

1 - The set of flag orbits R/G,

2 - The G_x-orbits of the extension sets for all $x \in \mathcal{T}(G, X)$

$$\bigcup_{x \in \mathcal{T}(G,X)} \pi_1^{-1}(x)/G_x, \tag{1}$$

3 - The G_y-orbits of the preimage sets for all $y \in \mathcal{T}(G, Y)$

$$\bigcup_{y \in \mathcal{T}(G,Y)} \pi_2^{-1}(y)/G_y, \tag{2}$$

The canonical correspondence between the objects in 1 and in 2 and between the objects in 1 and in 3 is characterized by the fact that objects correspond whenever they intersect nontrivially.

Theorem is called *Mackey's Theorem*, see Theorem 1.2.16 of kerber, if G was transitive on X and on Y. Now, we describe a main tool which plays an important role in Algorithm .

Definition 1 *Let* $\boldsymbol{G}$ *be a group acting on sets* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and let* $\boldsymbol{R} \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ *be a* $\boldsymbol{G}$*-invariant relation with* $\pi_2(\boldsymbol{R}) = \boldsymbol{Y}(\neq \emptyset)$*. We define a* μ*-function* $\mu : \boldsymbol{Y} \to \mathcal{P}(\boldsymbol{R})$ *(where* $\mathcal{P}(\boldsymbol{R})$ *is the power set of* $\boldsymbol{R}$*), with the following properties:*

1 - $\mu(y)$ is an orbit of G_y on $\pi_2^{-1}(y)$, and

2 - $\mu(y^g) = \mu(y)^g$ for all $g \in G$.

In other words, μ associates to every $y \in Y$ a non-empty (single) G_y-orbit $\mu(y) \subseteq \pi_2^{-1}(y)$ such that $\mu(y^g) = \mu(y)^g$ for all $g \in G$. These requirements on μ

ensure that μ identifies (or selects) one of the G_y-orbits on $\pi_2^{-1}(y)$ in a way that depends only on the G-orbit of y, but not on y itself. That is, if y was to be replaced by $z \in y^G$, then the orbit selected for z would be the image of the orbit selected for y under any isomorphism from y to z. Formally, if $z = y^g$ for $g \in G$, then the orbit selected for z is the g-image of the orbit selected for y. Such a G_y-orbit is called the *canonical orbit*. Such a μ-function can be realized by considering a canonical labeling map φ.

In the following two sections, we present two possible realizations of the μ-function, introduced by Read (1978) and Faradžev (1978) (independently), and by McKay (1998). Assuming that a μ-function is known, Theorem can be used to solve our main problem.

Theorem 2 *Let **G** be a group acting on two sets **X** and **Y**, and let $\boldsymbol{R} \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ be a **G**-invariant relation with $\pi_{\mathbf{2}}(\boldsymbol{R}) = \boldsymbol{Y}$. Assume that $\boldsymbol{T}(\boldsymbol{G}, \boldsymbol{X})$ and a defined μ-function are given. Then,*

$$T(G, Y) = \bigcup_{x \in T(G,X)} \pi_2\Big(\Big\{\ (x, y) \in T(G_x, \pi_1^{-1}(x)) \mid (x, y) \in \mu(y)\ \Big\}\Big) \tag{3}$$

is a transversal for the G-orbits on Y.

Theorem 2 describes the following two steps to solve the problem of constructing a transversal $\mathcal{T}(G, Y)$, when given a transversal $\mathcal{T}(G, X)$. First, consider the action of G_x for all $x \in T(G, X)$ on the set $\pi_1^{-1}(x)$ which results in a transversal $T(G, R)$. Second, we project by π_2 elements $(x, y) \in T(G, R)$ that are contained in the selected canonical G_y-orbit on $\pi_2^{-1}(y)$ using the definition of a suitable μ function satisfying the conditions of Definition . These two steps are called *lifting orbit* and *projecting orbit* steps, respectively. In practice, these two steps can be described by Algorithm 1.

A lgorithm 1 Two-Steps($\mathcal{T}$(G,X): an orbit transversal)

let $T(G, Y) = \emptyset$
for each $x \in T(G, X)$ **do**
compute $\pi_1^{-1}(x)$
compute $T(G_x, \pi_1^{-1}(x))$ **(lifting)**
for each $(x, y) \in T(G_x, \pi_1^{-1}(x))$ **do**
if $(x, y) \in \mu(y)$ **(projecting) then**
add y to $T(G, Y)$.
end if
end if
end it

For our convenience, given an element $(x, y) \in R \subseteq X \times Y$, we say that the element (x, y) is *accepted* if it passes the test in line (6) of Algorithm , and is *rejected* otherwise.

Orderly generation

In this section, we present a possible realization of a μ function satisfying the properties of Definition . Such a realization, called *orderly generation*, was introduced in the 1970's by Read (1978) and Faradžev (1978).

Let a group G act on two finite totally ordered sets X and Y, and let $R \subseteq X \times Y$ be a G-invariant relation. Recall that a canonical labeling map $\varphi : Y \to G$ maps an element $y \in Y$ to its orbit representative under the action of G, denoted by $y^{\varphi(y)}$. According to Read (1978) and Faradžev (1978), an element $y \in Y$ is called the canonical orbit representative, if y satisfies the following condition:

$$y \leq y^g, \quad \text{for all } g \in G. \tag{4}$$

In other words, the canonical orbit representative $y^{\varphi(y)}$ is the least element in the orbit y^G under the given total ordering of Y. In our case, we consider the *lexicographical ordering*. Further details about total ordering can be found, for instance, in (Al-Azemi, 2009; Betten *et al.*, 2006).

Theorem 3 *Let* $\boldsymbol{G}$ *be a group acting on two finite totally ordered sets* $\boldsymbol{X}$ *and* $\boldsymbol{Y}$*, and let* $\boldsymbol{R} \subseteq \boldsymbol{X} \times \boldsymbol{Y}$ *be a* $\boldsymbol{G}$*-invariant relation with* $\pi_2(\boldsymbol{R}) = \boldsymbol{Y}$*. Then, for any given* $\boldsymbol{y} \in \boldsymbol{Y}$*, let*

$$\mu(y) = \left[\left(\min\{\pi_2^{-1}(y^{\varphi(y)})\} \right)^{\varphi(y)^{-1}} \right]^{G_y}$$

where the minimum is taken with respect to the order on Y. Then, μ satisfies the conditions of Definition1. In particular, given $T(G, X)$, $T(G, Y)$ can be constructed by using Equation (1).

Prrot. We only need to show that the conditions of Definition are satisfied with this realization of μ. Then the proof is completed by Theorem 2.

Let us first show that $\mu(y)$ is a G_y-orbit on $\pi_2^{-1}(y)$. Clearly R is a G-invariant relation and if $G_y \leq G$, then G_y acts on R and R is a G_y-invariant relation. Since $\pi_2^{-1}(y) \subseteq R$ and G_y acts on R, we obtain that G_y acts also on $\pi_2^{-1}(y)$. Now, let $\min\{\pi_2^{-1}(y^{\varphi(y)})\} = (x_0, y^{\varphi(y)})$. Then by applying $\varphi(y)^{-1}$ on $(x_0, y^{\varphi(y)})$ we would get $(x_0^{\varphi(y)^{-1}}, y) \in \pi_2^{-1}(y)$. Hence the first property of μ holds.

It can easily be shown that $\varphi(y)\varphi(y^g)^{-1}$ maps y to y^g by the definition of a labeling map φ, and hence $\varphi(y)\varphi(y^g)^{-1}g^{-1} \in G_y$. Then

$$\begin{aligned}\mu(y^g) &= \left[\left(\min\{\pi_2^-1(y^{g\varphi(y^g)})\}\right)^{\varphi(y^g)^-1}\right]^{G_{y^g}}\\ &= \left[\left(\min\{\pi_2^-1(y^{\varphi(y)})\}\right)^{\varphi(y^g)^-1}\right]^{g^{-1}G_y g}\\ &= \left[\left(\min\{\pi_2^-1(y^{\varphi(y)})\}\right)^{(\varphi(y)^{-1}\varphi(y))\cdot\varphi(y^g)^{-1}g^{-1}}\right]^{G_y g}\\ &= \left(\left[\left(\min\{\pi_2^-1(y^{\varphi(y)})\}\right)^{\varphi(y)^{-1}}\right]^{G_y}\right)^g = \mu(y)^g.\end{aligned}$$

Thus μ satisfies the second property as well. Therefore, Algorithm constructs a transversal for the G-orbits on Y, given a transversal for the G-orbits on X.

We remark that, for any $y \in Y$, the mapping $\varphi(y)$ can be defined in several ways. One particular way is to map y to the lexicographically least (or greatest, if one prefers) element that is contained in the same G-orbit as y. Applications of orderly generation techniques appear in (Brinkmann, 1996; Dinitz *et al.*, 1994; Jϕrgensen, 1999; Meringer, 1999).

Canonical augmentation

In this section, we consider another realization of the μ-function introduced by McKay (1998). Note that most of the ideas introduced in this section is paraphrasing those of McKay (1998) in different language. The μ-function developed in this section is concerning our interests which are incidence structures that can be described as incidence matrices.

Let G be a group acting on a finite set X of incidence structures on m points and n blocks. The μ-function presented in this section does not depend on the lexicographical ordering as it does in orderly generation. It relies on a function φ: $X \to G$, which is the canonical labeling map, such that for all $x, y \in X$ we have $x^{\varphi(x)} = y^{\varphi(y)}$ if and only if x and y are contained in the same G-orbit on X. Such a function can be realized by the techniques of the partition backtracking, see (Mckay, 1977; Leon, 1991) for futher readings. Here, we assume that such a function is realized.

Recall that $\cup_{l=0}^m M_{m,n}^{(l)} = M_{m,n}$. For the sake of the simplicity, let X_l denote $M_{m,n}^{(l)}$ for $l = 0, 1, , m$ with $X = M_{m,n}$. Let $R_l \subseteq X_l \times X_{l+1}$ be a G-invariant relation for $l = 0, 1, , m - 1$.

For $x \in X_l$ and $y \in X_{l+1}$, we say that $(x, y) \in R_l$ if x is a pre-image of y, i.e. $x \prec y$. In particular, the set of pre-images of y is

$$\pi_2^{-1}(y) := \{(y - (E_i \otimes \mathrm{Row_i(y)), y) \in R_l \mid i \in row - supp(y)}\}.$$

We remark that $(y - (E_i \otimes \mathrm{Row_i(y)}) \in \mathrm{X_l}$ is the same as y with one non-zero row in y replaced by a zeros row.

Let $t := \min\{\mathrm{row - supp(y^{\varphi(y)})}\}$, and let $s := t^{\varphi(y)^{-1}}$. That is, t is the row index of the first (non-zero) row of the canonical matrix $y^{\varphi(y)}$, and s is the row index of the original row that was mapped onto t and is called the *canonical row*. Note that s is equivalent to t under the action of G_y.

Define $\mu(y) := \left\{(y - (E_i \otimes \mathrm{Row_i(y)), y) \in R_l \;\middle|\; E_i \in M_{m,1}, and i \in s^{G_y}}\right\}$. The G_y-orbit of s is called the *canonical orbit*. In simple words, we say that $(x, y) \in \mu(y)$ if $y \in X_{l+1}$ is an extension of $x \in X_l$ and that the last added row in y is contained in the canonical orbit s^{G_y}.

Theorem 4 *Let* $\boldsymbol{G}$ *be a group acting on two finite sets of incidence structures* $\boldsymbol{X_l}$ *and* $\boldsymbol{X_{l+1}}$ *for* $\boldsymbol{l = 0, 1, \ldots, m-1}$. *Let* $\boldsymbol{R_l \subseteq X_l \times X_{l+1}}$ *be a* $\boldsymbol{G}$*-invariant relation with* $\boldsymbol{\pi_2(R_l) = X_{l+1}}$. *Then, for any given* $\boldsymbol{y \in X_{l+1}}$, *let*

$$\mu(y) = \left[(y - (E_i \otimes \mathrm{Row_i(y)), y)}\right]^{\mathrm{G_y}} \tag{5}$$

where $i \in s^{G_y}$, and $E_i \in M_{m,1}$. Then, μ satisfies the conditions of Definition . In particular, given $T(G, X_l)$, $T(G, X_{l+1})$ can be constructed by using Equation (1).

Proof By Equation (5), $\mu(y)$ is a G_y-orbit on $\pi_2^{-1}(y)$. So, we only show that $\mu(y^g) = \mu(y)^g$ for all $g \in G$.

For any $g \in G$, it is clear that $(E_i \otimes \mathrm{Row_i(y^g))^{g^{-1}}} = \mathrm{E_i \otimes Row_i(y)}$. Then,

$$\begin{aligned}
\mu(y^g) &= \left[(y^g - (E_j \otimes \mathrm{Row_j(y^g)), y^g)}\right]^{\mathrm{G_{yg}}} \\
&= \left[(y^g - (E_j \otimes \mathrm{Row_j(y^g)), y^g)}\right]^{\mathrm{g^{-1}G_{yg}}} \\
&= \left[(y - (E_j \otimes \mathrm{Row_j(y^g))^{g^{-1}}, y)}\right]^{\mathrm{G_{yg}}} \\
&= \left(\left[(y - (E_i \otimes \mathrm{Row_i(y)), y)}\right]^{\mathrm{G_y}}\right)^{\mathrm{g}} = \mu(\mathrm{y})^{\mathrm{g}}.
\end{aligned}$$

Thus Equation (1) can be employed using the μ-function realized in Equation () to construct a transversal $\mathcal{T}(G, X_{l+1})$, given a transversal $\mathcal{T}(G, X_l)$.

An application of canonical augmentation

In the literature ((cf. Canfield & McKay, 2005)), authors have been interested in classifying all distinct $\{0, 1\}$ matrices on m rows and n columns with a constant row sum and a constant column sum. In this section, we classify all such non-isomorphic matrices, with $m, n \leq 12$, using Algorithm with the realized μ-function of Equation (3).

Let r and c be two non-negative integers representing row sum and column sum, respectively, for a given incidence matrix in $M_{m,n}$. It is clear that $rm = cn$.

For $l = 0, 1, , m$, define

$\mathcal{A}_l^r := \{A \in M_{m,n}^{(l)} | \text{row} - \text{sum}_i(A) = r \text{ for all } i \in \text{row} - \text{supp}(A)\}$,

$\mathcal{B}^c := \{A \in M_m, n | \text{col} - \text{sum}_j(A) \leq c \text{ for all } j \in \mathcal{C}\}$,

$\mathcal{D}_l := \{A \in M_{m,n} | A \in \mathcal{A}_l^r \cap \mathcal{B}^c\}$.

Note that we consider the search space $\mathcal{D} := \cup_{l=0}^{m} \mathcal{D}_l$ where the search target is $\mathcal{D}_m$. Matrices that are in $\mathcal{D}_m$ are called *feasible solutions*, while matrices that are in $\mathcal{D}_l$ are called *partial solutions*, for $l = 0, 1, , m-1$. Let $N(m, r; n, c)$ denote the number of all non-isomorphic $m \times n$ $\{0, 1\}$ matrices with row sum r and column sum c for each row and column. Equivalently, $N(m, r; n, c)$ is the number of non-isomorphic *semiregular bipartite* graphs with m vertices of degree r and n vertices of degree c.

Let $G = S_m \times S_n$ act on $\mathcal{D}_l$ and on $\mathcal{D}_{l+1}$ and let G act coordinate-wise on $\mathcal{D}_l \times \mathcal{D}_{l+1}$. Let $R_l \subseteq \mathcal{D}_l \times \mathcal{D}_{l+1}$ be a G-invariant relation such that $R_l = \{(A, B) \in \mathcal{D}_l \times \mathcal{D}_{l+1} \mid A \prec B\}$ for $l = 0, 1, , m-1$. In this language, we wish to construct a transversal $\mathcal{T}(G, \mathcal{D}_{l+1})$, given a transversal $\mathcal{T}(G, \mathcal{D}_l)$ by using the lifting orbit and projecting orbit steps.

First, the lifting orbits step suggests that for every $A \in \mathcal{T}(G, \mathcal{D}_l)$, for $l = 0, 1, , m-1$, we compute the corresponding extension set $\pi_1^{-1}(A)$ and the stabilizer group G_A which are needed to construct a transversal $\mathcal{T}(G_A, \pi_1^{-1}(A))$. Thus, a transversal $\mathcal{T}(G, R_l)$ is constructed. The second step is the projecting orbits step where we add an extension of A, say B, to a transversal $\mathcal{T}(G, \mathcal{D}_{l+1})$ only if

$$(A, B) \in \mathcal{T}(G_A, \pi_1^{-1}) \cap \mu(B), \tag{4}$$

where $\mu(B)$ is defined as in Equation (3). We remark that the μ-condition in Equation (4) guarantees that all augmented partial solutions must be canonical in order to be added to $\mathcal{T}(G, \mathcal{D}_{l+1})$ for $l = 0, 1, , m-1$. Once $\mathcal{T}(G, \mathcal{D}_m)$ has been constructed, the search algorithm stops and the desired results have been achieved.

We remark that Algorithm and the two steps described previously were presented in a breadth-first search (**BFS**). That is, construct all elements of $\mathcal{T}(G, \mathcal{D}_1)$, then construct all elements of $\mathcal{T}(G, \mathcal{D}_2)$, etc. But because of the fact that the augmentation procedure of elements $A \in \mathcal{T}(G, \mathcal{D}_l)$ depends only on A itself and not on other elements of $\mathcal{T}(G, \mathcal{D}_l)$, we can do a depth-first search (**DFS**) rather than a BFS.

One advantage of this algorithm is to avoid the case where some of the intermediate sets $\mathcal{T}(G, \mathcal{D}_l)$ get very large to store. Another one is that, we can partition the search between as many machines as are available. This can be done using a DFS, since the recursive step is just to augment the most recent produced element at all times.

In particular, we start the search for $l = 0$ with $A \in \mathcal{T}(G, \mathcal{D}_0)$, which is the unique matrix whose entries are all zeros, to construct all elements of $\mathcal{T}(G, \mathcal{D}_1)$ as described above. Then, we construct all elements of $\mathcal{T}(G, \mathcal{D}_2)$ from those of $\mathcal{T}(G, \mathcal{D}_1)$. When we reach elements of $\mathcal{T}(G, \mathcal{D}_m)$, the search is completed and a complete system of representatives has been constructed.

A modified version of Algorithm was applied to find all non-isomorphic matrices in $\mathcal{D}_m$ for $m \leq 12$ in the following way. We carry out a row-by-row backtrack search over all matrices contained in $\mathcal{D}$. We start with the matrix whose elements are all zeros. In order, we construct one row at a time. Each constructed row must have r ones and the columns must have less or equal c ones at all times. Once a row has been constructed, we accept it if it was canonical in the sense of Equation (3) and the search goes on. Otherwise, we reject it and we backtrack. When row m is accepted, a new isomorphism class has been determined and is added to the solutions of the problem at hand.

The search was done on two personal PCs with *Linux operating system*. The results of this search are presented in Table . We remark that for some parameters, $N(m, r; n, c)$ need not be computed explicitly. It is clear that $N(m, r; n, c) = N(m, n - r; n, m - c)$, by complementation. Moreover, by symmetry we have $N(m, r; n, c) = N(n, c; m, r)$. Up to isomorphism it is always true that $N(m, 0; n, 0) = 1$ (the all zeros matrix), $N(m, 1; m, 1) = 1$ (the identity matrix), and $N(m, n; n, m) = 1$ (the all ones matrix).

We conclude that apart from the new results in Table , another purpose of this paper is to present the theory of isomorph-rejection in a new fashion.

The number $N(12, 4; 12, 4) = 317, 579, 783$ is a new result which can be added to the integer sequence identified as A000513 in the on-line Encyclopedia of Integer Sequences. The search for this case was seperated on two machines and it was done in about 28 days CPU time. An equivalent result is that there are $317, 579, 783$ isomorphism classes of *bicolored quartic bipartite graphs*, where isomorphisms can not exchange the colors, see integer_sequences for further details.

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Table 1. Matrics in $M_{m,n}$ with row sum r and column sum c, upto isomorphishm

m	r	n	c	$N(m,r;n,c)$	m	r	n	c	$N(m,r;n,c)$
2	2	4	1	1	7	2	7	2	4
2	3	6	1	1	7	3	7	3	16
2	4	8	1	1	8	2	8	2	7
2	5	10	1	1	8	3	8	3	51
2	6	12	1	1	8	4	8	4	194
3	2	6	1	1	8	5	10	4	3,144
3	3	9	1	1	8	3	12	2	32
3	4	12	1	1	8	6	12	4	65,548
4	2	4	2	2	9	2	9	2	8
4	3	6	2	3	9	3	9	3	224
4	2	8	1	1	9	4	9	4	3,529
4	4	8	2	4	9	4	12	3	22,670
4	5	10	2	5	10	2	10	2	12
4	3	12	1	1	10	3	10	3	1,165
4	6	12	2	7	10	4	10	4	121,790
5	2	5	2	2	10	5	10	5	601,055
5	2	10	1	1	10	6	12	5	128,665,248
5	4	10	2	7	11	2	11	2	14
6	2	6	2	4	11	3	11	3	7,454
6	3	6	3	7	11	4	11	4	5,582,612
6	4	8	3	19	11	5	11	5	156,473,848
6	3	9	2	9	12	2	12	2	21
6	5	10	3	46	12	3	12	3	56,349
6	2	12	1	1	12	4	12	4	317,579,783
6	4	12	2	24					
6	6	12	3	132					

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نبذ التماثل : نظرية وتطبيق

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خلاصة

نصِف بتفصيل بعض خوارزميات التصنيف والتي تناسب مهام بحث عديدة مشتركة في نظرية التوافقات بشكل عام. سوف نركز على خوارزميتين للتصنيف الفَعال المنتظم وهما:

أولاً: خوارزمية التصنيف المنتظم، كانت بإكتشاف *Faradžev* و *Read* (كلٍ على حدا).

ثانياً: خوارزمية التصنيف المتراكم، وهي تنسب إلى *McKay*.

بعد ذلك، نقوم بتطبيق الطريقة اللاحقة لمكاي لتصنيف فئة من المصفوفات {0،1} ذات أ صف و ب عمود (بحيث أ، ب أقل من أو تساوي 12) شرط أن يكون مجموع كل صف وكل عمود ذو قيمة ثابتة. كنتيجة يوجد 317,551,913 {0،1} مصفوفة غير متطابقة ذات 12 صف و 12 عمود بحيث يكون مجموع كل صف وكل عمود 4.